42) Consider the parametric curve
\[ x = \frac{1 - t^2}{1 + t^2}, \quad y = \frac{2t}{1 + t^2}. \]
Describe this curve by polynomials in \(x, y,\) and \(t\). By eliminating \(t\), determine a polynomial in \(x\) and \(y\) describing the curve and use this result to identify the curve.

43) Consider a (2-dimensional) robot arm as depicted.
We want to find out the angles \(\theta_1, \theta_2\) to which the joints have to be set to move the hand to coordinates \((a, b)\). For simplification, assume the two arms have length \(l_1 = 3, l_2 = 4\).
To avoid using trigonometric functions, set \(s_i = \sin(\theta_i), \quad c_i = \cos(\theta_i) \quad (i = 1, 2)\). Then \(s_i^2 + c_i^2 = 1\).
Write down equations that determine \(a, b\) in terms of the variables \(c_1, s_1, c_2, s_2\). (You will have to use the formulas for \(\sin(\alpha + \beta)\) and \(\cos(\alpha + \beta)\).)

44) Let \(R = \mathbb{Q}[x, y, z]\) and \(I = (x^2 + yz - 2, y^2 + xz - 3, xy + z^2 - 5) \triangleleft R\). Show that \(x + I\) is a unit in \(R/I\) and determine \(x + I\)^{-1}.

45) a) Let \(F\) be a field and \(R = F[x_1, \ldots, x_n]\) and let \(I_1 = \langle f_1, \ldots, f_k \rangle \triangleleft R\) and \(I_2 = \langle h_1, \ldots, h_r \rangle \triangleleft R\) be two ideals. Let \(S = F[x_1, \ldots, x_n, t]\) (considering \(R \subseteq S\)) and set
\[ J = \langle t \cdot f_1, \ldots, t \cdot f_k, (1 - t) \cdot h_1, \ldots, (1 - t) \cdot h_r \rangle \triangleleft S. \]
Show that \(I_1 \cap I_2 = J \cap R\).

b) Let \(f = x^3z^2 + x^2yz^2 - xy^2z^2 - y^3z^2 + x^3y - x^2y^2 - xy^3\) and \(g = x^2z^4 - y^2z^4 + 2x^3z^2 - 2xy^2z^2 + x^4 - x^2y^2\). Compute \(\langle f \rangle \cap \langle g \rangle\).

c) Compute \(\gcd(f, g)\). (Hint: Show that \(\langle f \rangle \cap \langle g \rangle = \langle \text{lcm}(f, g) \rangle\).)

46) (Intended to illustrate the reason for having different monomial orderings.)
Let \(I = (x^5 + y^4 + z^3 - 1, x^3 + y^2 + z^2 - 1)\). Compute a Gröbner basis for \(I\) with respect to the \(\text{lex}\), \(\text{grlex}\), and \(\text{grevlex}\) orderings. Compare.
Repeat the calculations for \(I = (x^5 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1)\) (only one exponent changed!)

47) Let \(f: \mathbb{R}^n \rightarrow \mathbb{R}\). A point \(x \in \mathbb{R}^n\) is called a critical point of \(f\), if \(\frac{\partial f}{\partial x_i}(x) = 0\). (Cf. Calculus 3.)
Determine all critical points of the function
\[ f(x, y) = (x^2 + y^2)^3 - 4x^2y^2 \]
Note: this curve is the “four-leaved flower” \(r = \sin(2\theta)\) in polar coordinates. Is there a geometric interpretation of the critical points?