1) In this problem, we want to show that a subgroup of $S_6$ of index 6 must be isomorphic to $S_5$:
   a) Using the fact that $A_6$ is simple (you don’t need to prove this), show that the only normal subgroups of $S_6$ are $\langle 1 \rangle$, $A_6$, and $S_6$.
   b) Let $U \leq S_6$ be a subgroup of index 6. Let $\varphi$ be the action of $S_6$ on the cosets of $U$. Using a), show that the kernel of $\varphi$ must be the trivial subgroup and conclude that $\varphi$ is an isomorphism $S_6 \rightarrow S_6$.
   c) Show that the image of $\varphi(U)$ must be $S_5$, conclude that $U \cong S_5$.

2) In this problem, we will show that the group of rotations of an icosahedron is isomorphic to $A_5$.
   a) Let $H$ be the group of rotations and reflections of an icosahedron and $G \leq H$ the subgroup of rotations. (We have proven before that $|H| = 120$ and $[H : G] = 2$.) Let $C$ be the set of pairs of opposite corners of the icosahedron. Let $\sigma$ be the permutation representation of $H$, afforded by the action on $C$. Show that $H\sigma \leq S_6$.
   b) Determine $\bigcap_{c \in C} \text{Stab}_H(c)$ and show that $\sigma$ is an isomorphism.
   c) Conclude (using problem 1) that $H \cong S_5$ and that $G \cong A_5$.

3) How many different possibilities (up to rotations only) are there to color the faces of an icosahedron with the colors red and green, coloring 10 faces red and 10 green?
   **Hint:** From problem 2 we know that the acting group is isomorphic $A_5$. Use your knowledge about the classes of $A_5$ to show that a set of elements of certain orders must yield class representatives.

4) Let $G$ be a group and $U \leq G$ with $[G : U] = 3$. Show that $G$ possesses a normal subgroup of index 2 or 3.
   **Hint:** Consider the action of $G$ on the cosets of $U$. What are the possible image groups. Then use the isomorphism theorem.

5) Classify the transitive permutation representations of $S_4$: What is the degree? Is it faithful? (Can you give images for the generators $(1, 2, 3, 4), (1, 2)$ for each representation?)
   You may use GAP for any calculations. Problems marked with a * are bonus problems.