50) Consider the following network with an irrational flow capacity $\phi = (\sqrt{5} - 1)/2$.

![Network Diagram]

Now consider the following sequence of flow increases (here $k$ indicates the iteration, starting with $k = 0$ and no flow):

1. Augment by $\phi^k$ along sdcbat.
2. Augment by $\phi^k$ along sbcdt.
3. Augment by $\phi^{k+1}$ along sdcbat.
4. Augment by $\phi^{k+1}$ along sabct.
5. Go back to step 1.

a) Show that this iteration always augments maximally along the chosen path.
b) Show that the iteration does not converge and keeps the flow level below 10 (while there clearly is a maximal flow of 201).
c) What would happen if we replace $\phi$ with a close rational number? Why does this not contradict the fact that the Ford-Fulkerson algorithm is converging?

51) Consider the following problem: We have an image (considered as a matrix whose entries are the pixel values) and we want to separate foreground and background of the image. For this we assigned to each pixel (e.g. based on color information) a value $a_i$ to lie in the foreground and $b_i$ to lie in the background. (Note that $i$ and $j$ here are a pixel number as a single number, not $x$/$y$ coordinates!) We also assign to neighboring pixels a separation penalty $p_{i,j}$ if not both lie in foreground or background. (Again, this might depend on difference of color values, or even on some preliminary information about the content.) We set $p_{i,j} = 0$ if the pixels $i$ and $j$ are not neighbors. The task now is to find a partition $A \cup B$ ($A$ being foreground, $B$ being background) of the pixels that minimizes the penalty function

$$\sum_{i \in A} b_i + \sum_{i \in B} a_i + \sum_{i \in A, j \in B} p_{i,j}$$

Construct a network graph whose vertices are the pixels, together with a special “source” and “sink” vertex such that a partition $A \cup B$ corresponds to a minimum cut in this graph.

52) Show that the number of partitions of $n$ with at most $m$ parts equals the number of partitions in which no part exceeds $m$.

53) Show that for $n \geq 5$ we have that $p(n) < F_n$ where $F_n$ is the $n$-th Fibonacci number.
54) Let \( \sigma(n) = \sum_{d \mid n} d \) be the sum of divisors of \( n \) and let \( F(t) = \sum_{n \geq 0} \sigma(n) t^n \) and \( G(t) = \sum_{n \geq 0} \sigma(n) t^n \) be the generating function for the partition numbers, respectively sum-of-divisors. Prove (ignoring convergence) that
\[
\frac{d}{dt} F(t) = G(t) F(t)
\]
and deduce that
\[
np(n) = \sum_{k=1}^{n} \sigma(k) p(n-k).
\]

55) A permutation \( \pi \in S_n \) is called connected if there does not exist a number \( k, 1 < k < n \) such that \( \pi \) maps \( \{1, 2, \ldots, k\} \) to itself. Let \( c_n \) be the number of connected permutations. Prove that
\[
\sum_{i=1}^{n} c_i(n-i)! = n!
\]
Setting \( F(t) = \sum_{n \geq 1} n! \) and \( G(t) = \sum_{n \geq 1} c_n \), show that \( 1 - G(t) = (1 + F(t))^{-1} \).

56) Consider a tableau of shape \( (n,n) \). Define a sequence \( a_k \) \( (1 \leq k \leq 2n) \) by setting \( a_k := i \) if \( k \) is in row \( i \) of the tableau, \( i = 1, 2 \). Use this to show that the number of tableaux of this shape is given by the Catalan Number \( C_{n+1} \).

57) Show that there are exactly \( C_n \) Young diagrams that “fit in the shape” (that is could be placed in a space given by the shape without exceeding the edges) \( (n-1, n-2, \ldots, 1) \).

58) A Ballot Sequence (also called Lattice Permutation or Yamanouchi Word) of length \( n \) is a sequence of this length, made from \( l \) symbols \( 1, \ldots, l \), so that for any prefix of the word the number of \( i \)'s occurring is at least as great as the number of \( i + 1 \)'s occurring. We denote by \( a_i \) the number of occurrences of \( i \) in the sequence and say that the sequence is of type \( \lambda = (a_1, a_2, \ldots, a_l) \vdash n \). For example, for \( n = 5 \) and \( l = 2 \) the following are the sequences of type \( \lambda = (3, 2) \):

\[
\begin{align*}
&1122, &1121, &1212, &1211, &12121.
\end{align*}
\]

Show that for \( \lambda \vdash n \) there are exactly \( f_\lambda \) ballot sequences of type \( \lambda \). (Hint: The row number of entry \( i \) denotes the vote cast by voter \( i \).)

59) For \( \lambda = (a_1, a_2, \ldots, a_l) \vdash n \) a Lattice Path of type \( \lambda \) is a sequence of \( n + 1 \) points \( 0 = v_0, v_1, \ldots, v_n = (a_1, a_2, \ldots, a_l) \in \mathbb{R}^l \) (where \( l = l(\lambda) \)) such that \( v_i+1 - v_i \) is a unit coordinate vector, and such that all \( v_i \) stay in the cone \( x_1 \geq x_2 \geq \cdots x_l \geq 0 \). For example for \( \lambda = (3, 2) \) the following 5 paths exist:

\[
\begin{align*}
&\text{Path 1} \quad \text{Path 2} \quad \text{Path 3} \quad \text{Path 4} \quad \text{Path 5}
\end{align*}
\]

Show that there are exactly \( f_\lambda \) lattice paths of type \( \lambda \). (Hint: Problem 58.)
60) Using the Hook formula, determine the number of tableaux of the shape $7^15^13^21$.

61) For all $\lambda \vdash 6$, determine $f_\lambda$ using the hook formula. Calculate $\sum_{\lambda \vdash 6} f_\lambda^2$.

62) a) Show that if a diagram has a hook of length $a + b$ it must have a hook of length $a$ or of length $b$. (Hint: Consider a zig-zag path between the ends of the hook that goes around the bottom edge of the diagram)

b) Show that if a diagram has a hook of length $ab$ it must have hooks of length $a$ and length $b$.

63) The numbers (1 to 100) of 100 prisoners are placed randomly in 100 closed boxes, one number per box. The boxes, which are also labelled with numbers 1 to 100, are lined up on a table in a room. One by one the prisoners are led to the room individually. Each prisoner may look into 50 boxes (and close them again afterwards), but must leave the room exactly as they found it. Unless every prisoner opens a box containing his own name (the probability of this being $2^{-100}$ if every prisoner opens boxes by random !) all prisoners will (the usual melodramatic ploy of this kind of problem) be executed, if they all find their own number they all go free.

They may not communicate with the other prisoners about their findings once the first prisoner has entered the room, but they have the possibility to plot a strategy in advance. Find a strategy that has a success probability of over 30%.
(Hint: What is the probability of a permutation on 100 points to have no cycle of length > 50?)

Upcoming class schedule: 11/4: Prof. Patel, 11/7,11/9,11/14: Prof. Wilson, 11/11: Cancelled