44) You are given the information that 2 is a primitive root modulo $3^2$, modulo $5^2$ and modulo $11^2$. Let $n = 3^{123} \cdot 5^{456} \cdot 11^{789}$. Without using a computer, determine an element $a$ such that the order of $a$ modulo $n$ is equal to $\lambda(n)$.

45) Show that 3 is a primitive root modulo 401 (which is prime) and determine the index $\text{ind}_3(19)$ using Shanks’ algorithm.

46) Verify that 1373653 is a strong pseudoprinde to base 2 and base 3. Is it prime?

48) One of the shortest talks ever was given at the 1903 meeting of the American Mathematical Society. Frank N. Cole went to the blackboard and calculated $2^{67} - 1$ as well as $193707721 \cdot 761838257287$. He sat down again without saying a word. The audience applauded, as a factorization of this number (that had been disproved to be a prime already in 1875 by E. Lucas) had been considered impossible for a long time.¹
a) Prove that $2^{67} - 1$ is not prime.

b) Prove that the two factors given are prime, and give a certificate for one of them.

49)* Consider the following list of fractions:

\[
\frac{17}{91}, \frac{78}{85}, \frac{19}{51}, \frac{23}{38}, \frac{29}{33}, \frac{77}{29}, \frac{95}{23}, \frac{77}{19}, \frac{1}{17}, \frac{11}{13}, \frac{13}{11}, \frac{15}{2}, \frac{1}{7}, \frac{55}{1}\]

and let $n := 2$. We now perform the following process (invented by J.H. Conway):

Multiply $n$ by the first number $a$ in the list, for which the product $n \cdot a$ will be an integer and let $n := n \cdot a$. Whenever $n = 2^e$ is a power of 2, we write down the exponent $e$.

This process can be automatized, for example, by the following sequence of GAP commands:

```
list:=[17/91, 78/85, 19/51, 23/38, 29/33, 77/29, 95/23, 77/19, 1/17, 11/13, 13/11, 15/2, 1/7, 55/1];
n:=2;
while true do
  n:=First(list*n,IsInt); # form products, first integer
  e:=LogInt(n,2); # truncated logarithm
  if 2^e=n then # is it a power of two?
    Print(e,"\n"); # yes, print and do linefeed
  fi;
od;
```

a) What list of exponents will be produced?

b) Can you prove a)? (This is a very hard problem!)

Problems marked with a * are bonus problems for extra credit.

¹Later, asked how long this factorization took him, Cole responded: “The sundays of three years”