18) Let \( S = \text{Span}(1, \cos(2x), \cos^2(x), \sin^2(x)) \leq C[-10, 10] \). Determine the dimension of \( S \).

19) Let \( V = \{ p \in \mathbb{P}_3 \mid p(1) = 0, p(-1) = 0 \} \). (\( \mathbb{P}_3 \) are the polynomials of degree up to 3.) You are given the information that \( \dim(V) = 2 \).
   a) Find two linearly independent elements in \( V \).
   b) Show that the elements found in a) must span \( V \).

20) Let \( V \) be a vector space and \( X = \{ x_1, \ldots, x_m \} \subset V \) be a set of vectors. Also assume that \( m \leq n \); we set \( Y = \{ x_1, \ldots, x_m \} \subset X \) the set of the first \( m \) vectors in \( X \). Show:
   a) If \( Y \) is linear dependent then \( X \) is linear dependent.
   b) If \( X \) is linearly independent then \( Y \) is linearly independent.

21) Let \( A = \begin{pmatrix} 1 & 4 & -1 & -17 \\ 6 & 24 & -1 & -107 \\ -3 & -12 & 2 & 52 \end{pmatrix} \).
   a) Compute bases for \( N(A) \), \( RS(A) \) and for \( CS(A) \).
   b) What is the rank of \( A \)?

22) Let \( A \in \mathbb{F}^{m \times n} \) a matrix. For each of the following statements a)-r), indicate whether it is true or false.
   (You do not need to give a proof.)
   a) If \( \text{rank}(A) = m \) then the system \( Ax = b \) has at least one solution for every \( b \in \mathbb{F}^m \).
   b) If \( \text{rank}(A) = n \) then the system \( Ax = b \) has at least one solution for every \( b \in \mathbb{F}^m \).
   c) If \( \text{rank}(A) = m \) then the system \( Ax = b \) has at most one (i.e. no solution or a unique solution) for every \( b \in \mathbb{F}^m \).
   d) If \( \text{rank}(A) = n \) then the system \( Ax = b \) has at most one (i.e. no solution or a unique solution) for every \( b \in \mathbb{F}^m \).
   e) If \( Ax = b \) then \( x \in RS(A) \).
   f) If \( Ax = b \) then \( b \in CS(A) \).
   g) If \( B \in \mathbb{F}^{m \times n} \) is row equivalent to \( A \) then \( \text{rank}(A) = \text{rank}(B) \).
   h) If \( B \in \mathbb{F}^{m \times n} \) is row equivalent to \( A \) then \( CS(A) = CS(B) \).
   i) If \( B \in \mathbb{F}^{m \times n} \) is row equivalent to \( A \) then \( RS(A) = RS(B) \).
   j) If \( B \in \mathbb{F}^{m \times n} \) is row equivalent to \( A \) then \( \dim(N(A)) = \dim(N(B)) \).
   k) If \( B \in \mathbb{F}^{m \times n} \) is row equivalent to \( A \) then \( N(A) = N(B) \).
   l) If \( B \in \mathbb{F}^{m \times n} \) is column equivalent to \( A \) then \( \dim(N(A)) = \dim(N(B)) \).
   m) If \( B \in \mathbb{F}^{m \times n} \) is column equivalent to \( A \) then \( N(A) = N(B) \).
   n) If \( B \in \mathbb{F}^{m \times n} \) is row equivalent to \( A \) then \( B \) is also column equivalent to \( A \).
   o) If the columns of \( A \) are linearly independent then the rows of \( A \) are linearly independent.
   p) If \( CS(A) = \mathbb{F}^m \) and \( m \leq n \) then \( RS(A) = \mathbb{F}^n \).
   q) \( CS(A^T) = CS(A) \).
   r) \( \dim(CS(A^T)) = \dim(CS(A)) \).

23) Let \( \mathcal{B} = \{ (1, 0, -5)^T, (2, 1, -8)^T, (1, 0, -4)^T \} \) and \( \mathcal{C} = \{ (20, -16, 1)^T, (19, -15, 0)^T, (-43, 34, 0)^T \} \).
   a) Determine \( \mathcal{C} | [\text{id}]_{\mathcal{B}} \) and \( \mathcal{C} | [\text{id}]_{\mathcal{B}} \).
   b) What is \( [(1, 2, 3)^T]_{\mathcal{B}} \), what is \( \mathcal{B} \ast (1, 2, 3)^T \).
   c) Determine \( [(\mathcal{C} \ast (1, 1, 0))]_{\mathcal{B}} \).
24*) Let \( p(x) = \frac{1}{2}(x^2 - 5x + 6) \), \( q(x) = -x^2 + 4x - 3 \), \( r(x) = \frac{1}{2}(x^2 - 3x + 2) \).

a) Show that \( \mathbb{B} = \{ p,q,r \} \) is a basis of \( V = \mathbb{P}_2 = \{ ax^2 + bx + c \mid a,b,c \in \mathbb{R} \} \).

b) Determine \( [1]_{\mathbb{B}} \), \( [1+x]_{\mathbb{B}} \) and \( [x+x^2]_{\mathbb{B}} \).

c) Consider the values of \( p \), \( q \) and \( r \) at 1, 2 and 3. Determine a polynomial \( f(x) \) such that \( f(1) = 5 \), \( f(2) = -1 \), \( f(3) = 4 \).

Problems marked with a * are bonus problems for extra credit.

**Sample Midterm Problems**

We will have the first Midterm on Friday, February 17. This midterm will cover the material up to this homework sheet. There will be 5 problems in total.

The following problems are (in addition to all the homework problems on the previous sheets) typical midterm problems. (This does not mean that every midterm problem will be exactly of a type as given here.)

1) Which of the following subsets of \( \mathbb{R}^3 \) form a subspace? Give a proof if they are, respectively a concrete counterexample of vectors which violate one of the conditions, if not.

a) \( \{(a, b, c)^T \mid a + b = 0 \} \)

b) \( \{(a, b, c)^T \mid a + b = 1 \} \)

c) \( \{(a, b, c)^T \mid a = 5, b = 7, a + b = 1 \} \)

2) Let \( V \) be a vector space with basis \( \mathbb{B} = \{ \mathbf{b}_1, \ldots, \mathbf{b}_n \} \).

Prove that the set \( \{ \mathbf{b}_1, \ldots, \mathbf{b}_{n-1} \} \) can not be a spanning set for \( V \).

3) Let \( A \in \mathbb{F}^{m \times n} \) a matrix. For each of the following statements indicate whether it is true or false. Each correct answer is worth 2 points, each incorrect answer is worth -1 points. You cannot get less than 0 points in this problem.

a) If the system \( A \mathbf{x} = \mathbf{b} \) has at a solution for every \( \mathbf{b} \in \mathbb{F}^m \) then \( \text{CS}(A) = \mathbb{F}^m \).

b) If the system \( A \mathbf{x} = \mathbf{b} \) has at a solution for every \( \mathbf{b} \in \mathbb{F}^m \) then \( \text{RS}(A) = \mathbb{F}^n \).

c) If the system \( A \mathbf{x} = \mathbf{b} \) has at a solution for every \( \mathbf{b} \in \mathbb{F}^m \) then the columns of \( A \) are linearly independent.

d) If \( A \mathbf{x} = \mathbf{b} \) then \( x \in \text{RS}(A) \).

e) If \( B \in \mathbb{F}^{m \times k} \) then \( \text{N}(A) \subseteq \text{N}(AB) \).

f) If \( B \in \mathbb{F}^{m \times k} \) then \( \text{N}(B) \subseteq \text{N}(AB) \).

g) If \( \text{rank}(A) = m \) then the rows of \( A \) are linearly independent.

4) Extend the set \( \{ 1 + x + x^2, 3 + 2x \} \) to a basis of \( \mathbb{P}_3 \).

5) Let \( \mathbb{B} = \{(1,0,1)^T, (0,1,3)^T \} \subset \mathbb{R}^3 \) and \( V = \text{Span}(\mathbb{B}) \). (You may assume without proof that \( \mathbb{B} \) and \( \mathcal{C} \) both are linearly independent.) Let \( \mathcal{C} = \{(1,2,7)^T, (3,4,15)^T \} \).

a) Show that \( \text{Span}(\mathcal{C}) = V \). Explain your reasoning.

b) Determine \( \mathbb{B} \| \text{id} \|_{\mathcal{C}} \) and \( \mathcal{C} \| \text{id} \|_{\mathbb{B}} \).