67) Suppose we are making pairs \((a, b)\) of measurements in some experiment and end up with the following six pairs:

\[
\begin{array}{cccccc}
19 & 22 & 6 & 3 & 2 & 20 \\
12 & 6 & 9 & 15 & 13 & 5
\end{array}
\]

a) Shift the data such that the average of the pairs becomes \((0, 0)\)

b) Determine the correlation matrix \(C \in \mathbb{R}^{2 \times 2}\).

c) Find an orthogonal matrix \(U\) such that \(U^{-1}CU\) is diagonal.

d) Determine principal components (i.e. new coordinates in which the correlation matrix is diagonal) for this data.

68) Let \(A = U\Sigma V^T\) be a singular value decomposition. Show that the rank of \(A\) is equal to the number of nonzero singular values in \(\Sigma\).

69\*) Suppose the singular values of an invertible matrix \(A \in \mathbb{R}^{n \times n}\) are \(\sigma_1 \geq \sigma_2 \geq \cdots \sigma_n\).

a) Show that for every \(v \in \mathbb{R}^n\), we have that \(|Av| \leq \sigma_1|v|\) (where \(|\cdot|\) is the standard inner product on \(\mathbb{R}^n\)), with equality attained for some vector \(v \neq 0\).

b) Show similarly, that for \(v \in \mathbb{R}^n\), we have that \(|A^{-1}v| \leq \frac{1}{\sigma_n}|v|\).

**Note:** This indicates that the number \(\kappa = \sigma_1/\sigma_n\) indicates how much vectors can change under multiplication by \(A\). \(\kappa\) is called the condition number of \(A\). A large condition number indicates that a matrix is close to being singular, and that small errors on a vector \(b\) can have a large impact on the solutions of \(Ax = b\).

70\*) Prove that if \(A\) and \(B\) are similar matrices, then

a) \(A\) and \(B\) have the same eigenvalues with the same arithmetic and the same geometric multiplicities.

b) For each eigenvalue \(\lambda\) and each \(i\) we have that \(\dim N(A - \lambda \cdot I)^i = \dim N(B - \lambda \cdot I)^i\).

71) Write down the Jordan Canonical form of a matrix \(A\) which has the following kernel dimensions:

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
6 & 11 & 14 & 17 & 18 & 18
\end{array}
\]

72) Why is the following alleged sequence of kernel dimensions for the eigenvalue \(\lambda\) of a linear transformation \(L\) impossible?

\[
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 4 \\
0 & 1 & 2 & 6 & 6
\end{array}
\]

73\*) How many different matrices in Jordan Canonical Form exist that have the characteristic polynomial \((x - 1)^5(x - 2)^2\) are there? Write them all down.

Problems marked with a * are bonus problems for extra credit.