55) Consider the column spaces $F^n$ and $F^m$ with the respective standard inner products. Show that for $A \in F^{n \times n}$:
   a) $\text{RS}(A) = N(\text{RS}(A))$.
   b) $\text{CS}(A) = N(\text{RS}(A^T))$. (Hint: Use that $\text{CS}(A) = \text{RS}(A^T)$.)

56) Calculate an orthonormal basis for $V = P_2 = \text{Span}(1, x, x^2)$ with respect to the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, dx$.

57) Corrected Let $V$ be a finite-dimensional inner product space over $\mathbb{R}$.
   a) Let $v \in V$. Show that the map $L: V \to \mathbb{R}$, $w \mapsto \langle w, v \rangle = \langle v, w \rangle$ is linear. (Such a map is called a “functional”.)
   b) Show that if $L: V \to \mathbb{R}$ is linear there exists a vector $v \in V$ such that $L(w) = \langle w, v \rangle$ for any $w \in V$. (Hint: If $B$ is a basis for $V$ and $S$ the standard basis for $\mathbb{R}$, consider $A = [L]_S$. You need to find $\overline{v}$ such that $[\overline{v}]_B \cdot G_B = A$.)

58') Corrected Let $V$ be the space of all complex valued differentiable functions $f$, such that $\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} f(x) \overline{f(x)} \, dx < \infty$ with the inner product $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx$. (This space is sometimes called $L^2$. It is essentially the inner product space which is used in quantum mechanics.)
   a) Show that for every $f \in V$ we have that $\lim_{x \to \infty} f(x) = 0$ and $\lim_{x \to -\infty} f(x) = 0$.
   b) Show that the operator $D: L \to L$, $f(x) \mapsto i \cdot f'(x)$ is self-adjoint. (Hint: integration by parts. Use $\overline{f'} = \overline{f}$.)
   c') (This is technically hard and might require more Analysis than you have seen) Can you give an example of a nonzero function in $V$?

59) Let $A, B \in F^{n \times n}$ be orthogonal matrices.
   a) Show that $\det(A) \in \{1, -1\}$.
   b) Show that $AB$ and $A^{-1}$ both are orthogonal. (Thus the orthogonal matrices form a group.)

60') The following set of data is known to have come from a process which decays exponentially:

<table>
<thead>
<tr>
<th>$t$</th>
<th>.2</th>
<th>.5</th>
<th>.8</th>
<th>1.1</th>
<th>1.4</th>
<th>1.7</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>3.6766</td>
<td>2.1631</td>
<td>1.1797</td>
<td>0.7326</td>
<td>0.3455</td>
<td>0.3332</td>
<td>0.0996</td>
</tr>
</tbody>
</table>

That is, $y$ and $t$ are related by the expression $y = Ce^{kt}$ where $k$ is negative. There are measurement errors in the data, and we want to determine $C$, and $k$ using a least-squares process. For this, we transform the equation $y = Ce^{kt}$ by using natural logarithms and obtain the expression $\log y = \log(Ce^{kt}) = \log C + kt$.

Let $z = \log y$, $a = \log C$. Then we have $z = a + kt$ which is the equation of a line, or a polynomial of degree 1. The data set can be used to obtain an overdetermined system if we use the transformed pairs $(t, \log y)$.

a) Determine the table of transformed values $t, \log y$.
   b) Find the coefficients $a, c$ of the least squares fit line $z = a + kt$ through this data set.
   c) What are the values of $C$ and $k$ for the original problem?

Problems marked with a * are bonus problems for extra credit.