The following problems all cover material from M229, which is a prerequisite to this course.

1) In each of the following cases write down (or explain why it does not exist) a system with 3 equations and 3 variables that has:
   a) No solutions
   b) Exactly one solution
   c) Exactly two solutions
   d) Infinitely many solutions

2) A polynomial of the form $y = a_0 + a_1x + a_2x^2 + a_3x^3$ passes through the points $(-1, -10)$, $(0, -5)$, $(2, 11)$.
   a) Write the augmented matrix for a linear system whose solutions is the coefficients $a_0, a_1, a_2, a_3$.
   b) Solve the system and give the general form of a polynomial of degree at most 3 that passes through these points.

3) Does the matrix $A = \begin{pmatrix} 18 & -16 & 0 \\ 20 & -18 & 0 \\ 80 & -80 & 2 \end{pmatrix}$ have $\lambda = 2$ as an Eigenvalue? If so describe the set of associated eigenvectors.

4) Let $A, B \in \mathbb{R}^{n \times n}$.
   a) Show that $(AB)^T = B^T A^T$.
   b) Show that if $AB$ is invertible if both $A$ and $B$ are invertible, and that $(AB)^{-1} = B^{-1}A^{-1}$.
   c) Show that $AB$ cannot be invertible if $B$ is not invertible. (Note: You cannot use part b) here!)

5) A matrix $A \in \mathbb{R}^{2 \times 2}$ transforms points $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$ by multiplication $x \mapsto A \cdot x$. Suppose that the triangle $P, Q, R$ is transformed by this function into the triangle $P', Q', R'$ as sketched.
   a) Write a system of linear equations that must be satisfied by the entries of the matrix $A$.
   b) Solve for the entries of $A$.
   c) Determine the image of the point $(2, 2)$ under multiplication by $A$. 

\[ P \quad \quad \quad P' \]
\[ Q \quad \quad \quad Q' \]
\[ R \quad \quad \quad R' \]
We define the following functions \( f, g, h \) on \([0, 5]\):

\[
\begin{align*}
  f(x) &= \begin{cases} 
    x + 1, & 0 \leq x < 1 \\
    x^2 + 2x, & 1 \leq x < 3 \\
    3, & 3 \leq x \leq 5 
  \end{cases} \\
  g(x) &= \begin{cases} 
    x^3, & 0 \leq x < 2 \\
    -2x + 1, & 2 \leq x \leq 5 
  \end{cases} \\
  h(x) &= \begin{cases} 
    3x^3 + x + 1, & 0 \leq x < 1 \\
    3x^3 + x^2 + 2x, & 1 \leq x < 2 \\
    x^2 - 4x + 3, & 2 \leq x < 3 \\
    -6x + 6, & 3 \leq x \leq 5 
  \end{cases}
\end{align*}
\]

a) Show that \( 4f(2) + 11g(2) = h(2) \).

b) Is \( 4f - 11g = h \) ?

c) Is \( f + 3g = h \) ?

d) Compute \( 2h - g \).

Problems marked with a * are bonus problems for extra credit.