1. Let $S = \{1, 2, 3\}$ and let $A = \{1\}$. What is the cardinality of the set $P(S) - P(A)$?

**Answer:** By definition,

$$P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

and

$$P(A) = \{\emptyset, \{1\}\}.$$ 

Hence,

$$P(S) - P(A) = \{\{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\},$$

which has cardinality 6.

2. Prove that if $x \in \mathbb{R}$ and $x > 2$, then $x^2 - x > 2$. (*Hint: it’s often helpful to get 0 on the right side of inequalities.*)

**Coming up with the proof:** Following the hint, the first thing to do is to subtract 2 from both sides of both inequalities, yielding $x - 2 > 0$ and $x^2 - x - 2 > 0$. Now we can factor the left hand side of the second one, and we see that the goal is to prove that

$$(x + 1)(x - 2) > 0.$$ 

Now, we’re given that $x - 2 > 0$. Moreover, $x > 2$ clearly implies that $x + 1 > 0$ as well, so the left hand side above is the product of two positive numbers, and hence positive. Having thought this through, now we can write the actual proof:

**Proof.** Suppose $x \in \mathbb{R}$ so that $x > 2$. Then certainly $x + 1 > 0$. Also, $x > 2$ is equivalent to saying that $x - 2 > 0$. Therefore,

$$0 < (x + 1)(x - 2) = x^2 - x - 2.$$ 

Subtracting 2 from both sides, we see that

$$x^2 - x > 2,$$

as desired.

3. For each $n \in \mathbb{N}$, let $A_n = [0, \frac{3}{n})$.

(a) What is $\bigcup_{n \in \mathbb{N}} A_n$?

**Answer:** Since each $A_n$ is contained in $A_1 = [0, 3)$, we can see immediately that

$$\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} [0, \frac{3}{n}) = [0, 3).$$

(b) What is $\bigcap_{n \in \mathbb{N}} A_n$?

**Answer:** Notice that no negative numbers are contained in any of the $A_n$’s. Moreover, any positive number is not contained in $A_n$ when $n$ is sufficiently large, so we can conclude that

$$\bigcap_{n \in \mathbb{N}} A_n = \bigcap_{n \in \mathbb{N}} [0, \frac{3}{n}) = \{0\}.$$
4. Give an example of an infinite set $S \subset P(\mathbb{Q})$ so that $S \neq P(\mathbb{Q})$.

**Answer:** There are many such sets. For example, we could let

$$S = \{\{1\}, \{2\}, \{3\}, \ldots\} = \{\{n\} : n \in \mathbb{N}\},$$

which is clearly infinite and is not all of $P(\mathbb{Q})$ since it doesn’t contain every subset of $\mathbb{Q}$; for example, it doesn’t contain the set $\{1/2\}$.

We could also let $S = P(\mathbb{N})$, which is definitely infinite but is not all of $P(\mathbb{Q})$; again, it doesn’t contain $\{1/2\}$.

5. Re-write each of the following statements in symbols. Please explain any notation you introduce (for example, if you want $E(x)$ to stand for the open sentence “$x$ is even”, then please say so explicitly).

(a) All positive integers are prime.

**Answer:** There are a few different ways to do this. One is to let $P(x)$ be the statement “$x$ is prime” and then to write

$$\forall x \in \mathbb{Z}, (x > 0) \Rightarrow P(x).$$

A sneakier way to go is to restrict the universe to only include the positive integers, yielding the statement

$$\forall x \in \mathbb{N}, P(x).$$

(b) Some positive real numbers are irrational.

**Answer:** Let $R(x)$ be the statement “$x$ is rational”. Then we can translate the given statement as

$$\exists x \in \mathbb{R}, (x > 0) \land (\sim R(x)).$$

(c) All integers are perfect squares.

**Answer:** This is more of a challenge. Note that if $x$ is an integer, then $x$ being a perfect square means that $x = y^2$ for some integer $y$. Therefore, we can translate the given statement as

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x = y^2$$

or, if we want to let $S(x, y)$ be the statement that $x$ is the square of $y$, then we have

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, S(x, y).$$

6. For each of the statements in problem 5, say whether the statement is true or false. Explain your answer.

**Answer:**

(a) This is clearly false, since 4 is a positive integer which is not prime.

(b) This is true, since $\sqrt{2}$ is a positive, irrational real number.

(c) This is false: $-1$ is an integer which is not a perfect square.

7. Consider the following statement: For $n \in \mathbb{N}$, if $3n^2 - 2n - 1$ is even, then $n$ is odd. Either prove the statement or give a counterexample which shows that it is false.

**Answer:** This statement is true. Here’s the proof:

**Proof.** I will prove the contrapositive of the given statement, namely that if $n$ is even then $3n^2 - 2n - 1$ is odd. To that end, suppose $n \in \mathbb{N}$ is even. That means $n = 2m$ for some integer $m$. But then

$$3n^2 - 2n - 1 = 3(2m)^2 - 2(2m) - 1 = 12m^2 - 4m - 1 = 2(6m^2 - 2m) - 1,$$

which is odd since $6m^2 - 2m$ is an integer. This concludes the proof. ∎
8. Suppose that \(x, y \in \mathbb{Z}\) are both divisible by 3. Prove that for any integers \(a\) and \(b\), the integer \(ax + by + 1\) is not divisible by 3.

Proof. Let \(x, y, a, b \in \mathbb{Z}\) so that \(x\) and \(y\) are multiples of 3. In other words, \(x = 3n\) and \(y = 3m\) for some \(n, m \in \mathbb{Z}\). Therefore,

\[
ax + by + 1 = a(3n) + b(3m) + 1 = 3(an + bm) + 1.
\]

Since \(an + bm\) is an integer, the above is 1 more than a multiple of 3 and hence cannot be a multiple of 3, as desired.

9. Let \(O = \{2n - 1 : n \in \mathbb{N}\}\) and let \(T = \{3m : m \in \mathbb{N}\}\). Write the set \(O \cap T\) in two different ways.

**Answer:** Of course there are many ways to do this. The key is to notice that \(O \cap T\) consists precisely of the odd multiples of 3. One way to specify this set is by listing the elements:

\[O \cap T = \{3, 9, 15, 21, \ldots\}\]

Notice that the above set consists of precisely those numbers which are 3 less than a multiple of 6, so we have

\[O \cap T = \{6n - 3 : n \in \mathbb{N}\}\]

A related way to write the set is to notice that \(O \cap T\) consists of those multiples of 3 which are not multiples of 6:

\[O \cap T = T - \{6n : n \in \mathbb{N}\}\]

10. For each of the following statements, convert it to symbols and then say whether the statement is a tautology, a contradiction, or neither and explain why.

(a) Either I won’t bring my umbrella or if it rains I’ll bring my umbrella.

**Answer:** Let \(U\) be the statement “I bring my umbrella” and let \(R\) be the statement “It rains”. Then we can translate the above statement as

\[(\sim U) \lor (R \Rightarrow U)\]

This is fact is a tautology, as we can see by writing out the truth table and noting that the statement is true regardless of the truth or falsity of \(U\) and \(R\):

<table>
<thead>
<tr>
<th>(R)</th>
<th>(U)</th>
<th>(\sim U)</th>
<th>(R \Rightarrow U)</th>
<th>((\sim U) \lor (R \Rightarrow U))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T)</td>
<td>(T)</td>
<td>(F)</td>
<td>(T)</td>
<td>(T)</td>
</tr>
<tr>
<td>(T)</td>
<td>(F)</td>
<td>(T)</td>
<td>(F)</td>
<td>(T)</td>
</tr>
<tr>
<td>(F)</td>
<td>(T)</td>
<td>(F)</td>
<td>(T)</td>
<td>(T)</td>
</tr>
<tr>
<td>(F)</td>
<td>(F)</td>
<td>(T)</td>
<td>(T)</td>
<td>(T)</td>
</tr>
</tbody>
</table>

(b) No matter whether it rains or not, I’m bringing my umbrella.

**Answer:** Let \(R\) and \(U\) be as above. Then the most reasonable translation of the above sentence is

\[(R \lor (\sim R)) \Rightarrow U\]

This is definitely neither a tautology nor a contradiction, since it’s true if \(U\) is true and false if \(U\) is false. We can see this by writing out the truth table:

<table>
<thead>
<tr>
<th>(R)</th>
<th>(U)</th>
<th>(\sim R)</th>
<th>(R \lor (\sim R))</th>
<th>((R \lor (\sim R)) \Rightarrow U)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T)</td>
<td>(T)</td>
<td>(F)</td>
<td>(T)</td>
<td>(T)</td>
</tr>
<tr>
<td>(T)</td>
<td>(F)</td>
<td>(T)</td>
<td>(F)</td>
<td>(F)</td>
</tr>
<tr>
<td>(F)</td>
<td>(T)</td>
<td>(T)</td>
<td>(T)</td>
<td>(T)</td>
</tr>
<tr>
<td>(F)</td>
<td>(F)</td>
<td>(T)</td>
<td>(T)</td>
<td>(F)</td>
</tr>
</tbody>
</table>