Math 318 Exam #1 Solutions

1. (a) Suppose \((f_n)\) and \((g_n)\) are two sequences of functions that converge uniformly on a subset \(A \subset \mathbb{R}\). Is it true that the sequence \((f_ng_n)\) converges uniformly on \(A\)? Prove or give a counterexample.

**Answer.** No. For each \(n\), let 
\[ f_n(x) = \frac{1}{x} \quad \text{and} \quad g_n(x) = \frac{1}{n} \]
on the interval \((0, 1)\). Then certainly \(f_n \to \frac{1}{x}\) uniformly (since it’s a constant sequence) and \(g_n \to 0\) uniformly (since the value of \(g_n\) is independent of \(x\)). However, I claim that the sequence 
\[ ((f_n g_n)(x)) = \left( \frac{1}{nx} \right) \]
does not converge uniformly. Notice that, for any given \(\epsilon > 0\) and any fixed \(x \in (0, 1)\), we can pick \(N > \frac{1}{\epsilon x}\) so that \(n \geq N\) implies 
\[ \left| \frac{1}{nx} - 0 \right| = \frac{1}{nx} \leq \frac{1}{N x} < \epsilon, \]
so the sequence \((1/(nx))\) converges to the zero function pointwise. However, for any fixed \(N \in \mathbb{N}\), we can choose \(x\) such that \(0 < x < 1/N\) so that 
\[ \left| \frac{1}{N x} - 0 \right| = \frac{1}{N x} > 1, \]
so the convergence cannot be uniform.

(b) If your answer to (a) was “yes”, does the result still hold if one of the sequences does not converge uniformly? If your answer to (a) was “no”, what additional assumptions on \((f_n)\) and \((g_n)\) will yield uniform convergence of \((f_n g_n)\)?

**Answer.** Notice that the limiting function \(1/x\) above was not bounded (though the limit of the \(g_n\) obviously is). Here’s the modified claim: Suppose \(f_n \to f\) uniformly and \(g_n \to g\) uniformly on \(A \subseteq \mathbb{R}\) and that both \(f\) and \(g\) are bounded. Then \((f_ng_n)\) converges uniformly on \(A\).

**Proof.** Let \(\epsilon > 0\). Suppose \(M_1 > 0\) is an upper bound for \(f\) (meaning \(|f(x)| \leq M_1\) for all \(x \in A\)) and that \(M_2\) is an upper
bound for $g$. Since $f_n \to f$ uniformly, there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies

$$|f_n(x) - f(x)| < 1$$

for all $x \in A$. Equivalently, $|f_n(x)| < |f(x)| + 1 \leq M_1 + 1$ for all $x \in A$.

In turn, since $g_n \to g$ uniformly, there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies

$$|g_n(x) - g(x)| < \frac{\epsilon}{2(M_1 + 1)}.$$

Finally, since $f_n \to f$ uniformly, there exists $N_3 \in \mathbb{N}$ such that $n \geq N_3$ implies

$$|f_n(x) - f(x)| < \frac{\epsilon}{2M_2}.$$

Now, if $N = \max\{N_1, N_2, N_3\}$ and $n \geq N$, we have

$$|f_n(x)g_n(x) - f(x)g(x)| = |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)|$$

$$\leq |f_n(x)g_n(x) - f_n(x)g(x)| + |f_n(x)g(x) - f(x)g(x)|$$

$$= |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)|$$

$$\leq (M_1 + 1)|g_n(x) - g(x)| + M_2|f_n(x) - f(x)|$$

$$< (M_1 + 1)\frac{\epsilon}{2(M_1 + 1)} + M_2\frac{\epsilon}{2M_2}$$

$$= \epsilon$$

for all $x \in A$, so we see that, indeed, $(f_n g_n) \to (fg)$ uniformly on $A$. \qed

2. (a) Let $(f_n)$ be a sequence of continuous functions. Suppose $f_n \to f$ uniformly on $A \subseteq \mathbb{R}$. Prove that

$$\lim_{n \to \infty} f_n(x_n) = f(x)$$

for all $x \in A$ and all sequences $(x_n)$ in $A$ converging to $x$.

**Proof.** Fix $x \in A$ and let $(x_n)$ be a sequence converging to $x$. Let $\epsilon > 0$. The fact that $f_n \to f$ uniformly on $A$ implies that there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies

$$|f_n(y) - f(y)| < \epsilon/2$$
for all \( y \in A \).

In turn, since \( f_n \to f \) uniformly and each \( f_n \) is continuous, the limit function \( f \) is also continuous. Hence, by Theorem 4.3.2(iv), 
\[
\lim_{n \to \infty} f(x_n) = f(x),
\]
so there exists \( N_2 \in \mathbb{N} \) such that \( n \geq N_2 \) implies
\[
|f(x_n) - f(x)| < \epsilon / 2.
\]

Therefore, if \( N = \max \{N_1, N_2\} \) and \( n \geq N \), then
\[
|f_n(x_n) - f(x)| = |f_n(x_n) - f(x_n) + f(x_n) - f(x)| \\
\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \\
< \epsilon / 2 + \epsilon / 2 \\
= \epsilon.
\]

Since the choice of \( \epsilon > 0 \) was arbitrary, we see that \( \lim_{n \to \infty} f_n(x_n) = f(x) \). In turn, since the choice of \( x \in A \) was arbitrary, we see that this is true for all \( x \in A \), as desired.

(b) Is the converse true?

**Answer.** No. Consider the functions \( f_n(x) = \frac{x}{n} \) on all of \( \mathbb{R} \). Then \( (f_n(x)) \to 0 \), but not uniformly. However, the conclusion of part (a) will hold. To see this, let \( x \in \mathbb{R} \) and \( (x_n) \to x \). Let \( \epsilon > 0 \).

Since \( (x_n) \to x \), there exists \( N_1 \in \mathbb{N} \) such that \( n \geq N_1 \) implies \( |x_n - x| < 1 \), meaning that \( |x_n| < |x| + 1 \). Also, we can certainly pick \( N_2 \) such that \( N_2 > \frac{|x| + 1}{\epsilon} \).

Therefore, if \( N = \max \{N_1, N_2\} \) and \( n \geq N \), we see that
\[
|f_n(x_n) - f(x)| = |f_n(x_n) - 0| = \frac{|x_n|}{n} \leq \frac{|x| + 1}{n} \leq \frac{|x| + 1}{N_2} < \epsilon.
\]

Since the choice of \( \epsilon > 0 \) was arbitrary, we see that, indeed, \( \lim_{n \to \infty} f_n(x) = f(x) \). Since the choice of sequence \( (x_n) \to x \) was arbitrary, the same holds for any sequence converging to \( x \). Finally, since the choice of \( x \) was arbitrary, this is true for any \( x \in \mathbb{R} \).

3. Let \( f_n \) be a monotone increasing, continuous function on \([0, 1]\) for each \( n \in \mathbb{N} \). Suppose \( f(x) = \sum_{n=0}^{\infty} f_n(x) \) converges for every \( x \in [0, 1] \). Show that the function \( f \) is continuous on \([0, 1]\).
Proof. Let $\epsilon > 0$. The goal is to show that the series satisfies the hypotheses of the Cauchy Criterion. To that end, note that, since $\sum f_n(0)$ converges, there exists $N_0 \in \mathbb{N}$ such that $n > m \geq N_0$ implies

$$|f_{m+1}(0) + \ldots + f_n(0)| < \epsilon/2.$$  

Likewise, since $\sum f_n(1)$ converges, there exists $N_1 \in \mathbb{N}$ such that $n > m \geq N_1$ implies

$$|f_{m+1}(1) + \ldots + f_n(1)| < \epsilon/2.$$  

Let $N = \max\{N_0, N_1\}$ and let $n > m \geq N$. Since each $f_i$ is increasing, the function $f_{m+1} + \ldots + f_n$ is also increasing, so Lemma 3.1, stated below, implies that

$$|f_{m+1}(x) + \ldots + f_n(x)| \leq |f_{m+1}(0) + \ldots + f_n(0)| + |f_{m+1}(1) + \ldots + f_n(1)|$$

$$< \epsilon/2 + \epsilon/2$$

$$= \epsilon$$

for any $x \in [0, 1]$. Therefore, the series $\sum f_n$ satisfies the hypotheses of the Cauchy Criterion for Uniform Convergence, so we see that $\sum f_n$ converges uniformly.

Since each $f_n$ is continuous, Theorem 6.4.2 implies that $f$ is also continuous. \hfill \Box

Lemma 3.1. Suppose $g : [a, b] \to \mathbb{R}$ is monotone increasing. Then

$$|g(x)| \leq |g(a)| + |g(b)|$$

for any $x \in [a, b]$.

Proof. The fact that $g$ is increasing implies that

$$g(a) \leq g(x) \leq g(b)$$

for any $x \in [a, b]$. Therefore,

$$|g(x)| \leq \max\{|g(a)|, |g(b)|\} \leq |g(a)| + |g(b)|,$$

as desired. \hfill \Box
4. Consider a series $\sum_{n=1}^{\infty} a_n$. Let $S_n = \sum_{k=1}^{n} a_k$ be the $n$th partial sum and define

$$\sigma_n = \frac{\sum_{k=1}^{n} S_k}{n}.$$  

We say that the series $\sum_{n=1}^{\infty} a_n$ is Cesaro summable to $L$ if $\lim_n \sigma_n = L$. A consequence of HW 4 Problem #5 from last semester is that if $\sum a_n = L$, then the series is Cesaro summable to $L$.

(a) Show that the series $\sum_{n=1}^{\infty} (-1)^{n+1}$ is Cesaro summable to $\frac{1}{2}$.

(You showed on HW 3 Problem #5 that this is also the Abel sum of the series.)

**Proof.** Clearly,

\[
S_1 = 1 \\
S_2 = 1 - 1 = 0 \\
S_3 = 1 - 1 + 1 = 1 \\
S_4 = 1 - 1 + 1 - 1 = 0 \\
\vdots \\
S_k = \frac{1}{2} + \frac{(-1)^{k+1}}{2}
\]

so

$$\sigma_n = \frac{\sum_{k=1}^{n} S_k}{n} = \frac{\sum_{k=1}^{n} \left(\frac{1}{2} + \frac{(-1)^{k+1}}{2}\right)}{n} = \frac{n}{2} + S_n \frac{1}{2n} = \frac{1}{2} + S_n \frac{1}{2n}.$$  

Since $S_n$ is always either 0 or 1, the term $\frac{S_n}{2n}$ certainly goes to zero as $n \to \infty$, so

$$\lim_{n \to \infty} \sigma_n = \lim_{n \to \infty} \left(\frac{1}{2} + \frac{S_n}{2n}\right) = \frac{1}{2}.$$  

Hence, the series is indeed Cesaro summable to $\frac{1}{2}$. \qed

(b) Does there exist a series $\sum_{n=1}^{\infty} a_n$ that is Abel summable but not Cesaro summable?

**Answer.** Yes. Consider $a_n = (-1)^n n$. Then on $[0,1)$ we have that

$$\sum_{n=1}^{\infty} (-1)^n n x^n = \int_0^1 \sum_{n=0}^{\infty} (-1)^n n x^n - \int_0^1 \sum_{n=0}^{\infty} (-1)^n n x^n = \int_0^1 \frac{1}{1 + x} - \int_0^1 \frac{1}{1 + x} = \frac{-x}{(1 + x)^2}.$$  

\[
\sum_{n=1}^{\infty} (-1)^n n x^n = x \sum_{n=1}^{\infty} (-1)^n n x^{n-1} = x \frac{d}{dx} \left( \sum_{n=0}^{\infty} (-1)^n x^n \right) = x \frac{d}{dx} \left( \frac{1}{1 + x} \right) = \frac{-x}{(1 + x)^2}.\]
Therefore, since the function on the right is continuous at 1, we see that
\[
\lim_{x \to 1^-} \sum_{n=1}^{\infty} (-1)^n nx^n = \lim_{x \to 1^-} \frac{-x}{(1 + x)^2} = -\frac{1}{4}
\]
is the Abel sum of the series.

However, the partial sums look like

\[
\begin{align*}
S_1 &= -1 \\
S_2 &= -1 + 2 = 1 \\
S_3 &= -1 + 2 - 3 = -2 \\
S_4 &= -1 + 2 - 3 + 4 = 2 \\
&\vdots \\
S_{2k-1} &= -k \\
S_{2k} &= k
\end{align*}
\]

Therefore,
\[
\sigma_n = \begin{cases} 
0 & \text{if } n \text{ is even} \\
-\frac{n+1}{2n} & \text{if } n \text{ is odd.}
\end{cases}
\]

The even terms go to zero, but the odd terms go to \(-1/2\), so \(\lim \sigma_n\) does not exist, so the series is not Cesaro summable.

5. Let \(\sum_{n=0}^{\infty} a_n x^n\) be a power series with each \(a_n \geq 0\). Suppose that the radius of convergence is 1, so that the power series defines a function \(f(x) = \sum_{n=0}^{\infty} a_n x^n\) at least on \((-1, 1)\).

Prove that the power series converges at \(x = 1\) (meaning \(f(1)\) makes sense) if and only if \(f\) is bounded on \([0, 1)\).

\textbf{Proof.} \((\Rightarrow)\) If the power series converges at \(x = 1\), then Abel’s Theorem implies that \(f\) is continuous on the compact set \([0, 1]\) and, therefore, bounded on that set. Hence, \(f\) is bounded on \([0, 1)\).

\((\Leftarrow)\) On the other hand, suppose \(f\) is bounded on \([0, 1)\). My goal is to show that the sequence of partial sums \(s_k(1) = \sum_{n=0}^{k} a_n\) is bounded above; since each \(a_n \geq 0\), the sequence \((s_k(1))\) is monotone increasing, so boundedness plus the Monotone Convergence Theorem will imply convergence.
To see that the $s_k(1)$ are bounded, let $M > 0$ be an upper bound for $f(x) = \sum a_n x^n$ on $[0,1)$, meaning that $|f(x)| \leq M$ for all $x \in [0,1)$.

Now, I claim that $M$ is an upper bound for $s_k(1)$ for all $k \in \mathbb{N}$. Suppose not. Then there exists $K \in \mathbb{N}$ such that $s_K(1) > M$. But then, since $s_K(x) \leq f(x) \leq M$ for all $x \in [0,1)$, we see that

$$|s_K(1) - s_K(x)| > s_K(1) - M$$

for any $x \in [0,1)$.

But then this means that $s_K$ cannot possibly be continuous at $x = 1$, since there is no $\delta$ that will work for $\epsilon = s_K(1) - M > 0$. On the other hand, $s_K$ is clearly continuous since it is just a polynomial. From this contradiction, then, we can conclude that $M$ is indeed an upper bound for $s_k(1)$ for every $k$ and, therefore, that the sequence $(s_k(1))$ converges, which is exactly what it means to say that the power series converges at $x = 1$.  \qed