1. Find the volume of the solid generated by revolving the following region around the line \( x = -1 \). The region is in the first quadrant and bounded above by the parabola \( y = x^2 \), below by the \( x \)-axis, and on the right by the line \( x = 1 \).

**Answer:** We’re revolving the region bounded by the \( x \)-axis and the purple and red curves around the vertical blue line.

We could use either the washer or the shell methods for this problem. For the washer method, the cross-sectional washer will be horizontal, with outer radius determined by the red line and the inner radius determined by the purple curve. The area of the cross section changes as we change \( y \), so the cross-sectional area should be a function of \( y \), so we re-write the purple curve as the graph of a function of \( y \): \( x = \sqrt{y} \). Specifically, the cross-sectional area is given by

\[
A(y) = \pi(2)^2 - \pi(\sqrt{y} - (-1))^2 = 4\pi - \pi(y+2\sqrt{y}+1) = 4\pi - \pi y - 2\pi\sqrt{y} = 3\pi - \pi y - 2\pi\sqrt{y}.
\]

Since the minimum \( y \)-value contained in the region is \( y = 0 \) and the maximum \( y \)-value contained in the region is \( y = 1 \), we should integrate from 0 to 1. Therefore, the volume of the solid will be

\[
\int_0^1 A(y) \, dy = \int_0^1 (3\pi - \pi y - 2\pi\sqrt{y}) \, dy
\]

\[
= \left[ 3\pi y - \frac{\pi y^2}{2} - 2\pi \frac{2}{3} y^{3/2} \right]_0^1
\]

\[
= 3\pi - \frac{\pi}{2} - \frac{4\pi}{3}
\]

\[
= \frac{7\pi}{6}.
\]

On the other hand, if we want to use the shell method, our cylindrical shell will have vertical sides going from the \( x \)-axis to the purple curve. The size of the shell changes as we change \( x \),
and for a given \( x \) the radius of the shell will be \( x - (-1) = x + 1 \) and the height of the shell will be \( x^2 \). Therefore, the volume of the region will be

\[
\int_0^1 2\pi(x + 1)(x^2)\,dx = \int_0^1 2\pi(x^3 + x^2)\,dx
\]

\[
= 2\pi \left[ \frac{x^4}{4} + \frac{x^3}{3} \right]_0^1
\]

\[
= 2\pi \left[ \frac{1}{4} + \frac{1}{3} \right]
\]

\[
= \frac{7\pi}{6}.
\]

Either way, we see that the volume of the solid is \( \frac{7\pi}{6} \).

2. Find the volume of the solid generated by revolving the region bounded by the curves \( y = \sqrt{x} \) and \( y = x - 2 \) and the line \( x = 0 \) around the \( y \)-axis.

**Answer:** We’re revolving the region contained between the red curve, the blue curve, and the \( x \)-axis around the \( y \)-axis.

Since we’re revolving around the \( y \)-axis, if we used the washer method, our washers would be horizontal, which would make things slightly complicated since for part of the time they would be solid disks and for part of the time they would be actual washers with holes in the middle.

Instead, let’s use the shell method, since our cylindrical shells will be vertical with sides running from the blue curve to the red line. Therefore, the shell with radius \( x \) will have height

\[
\sqrt{x} - (x - 2) = \sqrt{x} - x + 2.
\]
Since the curves intersect when $\sqrt{x} = x - 2$, which is to say when $x = 4$, we know that the region runs from $x = 0$ to $x = 4$. Therefore, the volume of the solid is given by

$$
\int_0^4 2\pi x (\sqrt{x} - x + 2) \, dx = 2\pi \int_0^4 \left( x^{3/2} - x^2 + 2x \right) \, dx
$$

$$
= 2\pi \left[ \frac{2}{5} x^{5/2} - \frac{x^3}{3} + x^2 \right]_0^4
$$

$$
= 2\pi \left[ \frac{64}{5} - \frac{64}{3} + 16 \right]
$$

$$
= 2\pi \left[ -\frac{128}{15} + \frac{240}{15} \right]
$$

$$
= \frac{224\pi}{15}.
$$

3. Find the volume of the solid generated by revolving the region enclosed by the graph of $y = e^{-x^2}$ and the lines $y = 0$, $x = 0$, and $x = 2$ around the $y$-axis.

**Answer:** We’re revolving the region bounded by the axes, the red curve, and the blue line around the $y$-axis.

Since the axis of rotation is vertical, cross-sectional washers would be horizontal. This means, if we wanted to use the washer method we’d need to split the computation up into two integrals: one where the outer edge of the washer is the blue line, and one where the outer edge is the red curve.

Instead, we can do things with a single integral by using the shell method. Since the axis of rotation is vertical, our cylindrical shell will have vertical sides running from the $x$-axis up to the red curve. Since the shell changes as $x$ changes, we should integrate with respect to $x$. The shell at $x$ will have radius $x - 0 = x$ and height $e^{-x^2} - 0 = e^{-x^2}$. Therefore, the volume
of the solid is
\[ \int_{0}^{2} 2\pi(x) \left( e^{-x^2} \right) \, dx = \int_{0}^{2} 2\pi x e^{-x^2} \, dx. \]
If we let \( u = -x^2 \), then \( du = -2x \, dx \), so the above integral can be written as
\[
-\pi \int_{0}^{4} -2xe^{-x^2} \, dx = -\pi \int_{0}^{4} e^u \, du
\]
\[
= -\pi \left[ e^u \right]_{0}^{4}
\]
\[
= -\pi \left[ e^{-4} - e^{0} \right]
\]
\[
= -\pi \left[ \frac{1}{e^4} - 1 \right]
\]
\[
= \pi - \frac{\pi}{e^4}.
\]
Therefore, the volume of the solid is \( \pi - \frac{\pi}{e^4} \approx 3.08 \).