1. Find the derivatives of the following functions.

(a) \( f(x) = \ln(\arctan(x)) \)

**Answer:** Using the Chain Rule,

\[
 f'(x) = \frac{1}{\arctan(x)} \frac{d}{dx} (\arctan(x))
\]

Remembering that the derivative of \( \arctan(x) \) is \( \frac{1}{1+x^2} \), the above becomes

\[
 f'(x) = \frac{1}{\arctan(x)} \frac{1}{1+x^2}.
\]

(b) \( g(x) = \frac{\sin(x)}{\cos^2(x)} \)

**Answer:** The hard way to solve this is using the Quotient Rule:

\[
 g'(x) = \frac{\cos^2(x) \cdot \frac{d}{dx} (\sin(x)) - \sin(x) \cdot \frac{d}{dx} (\cos^2(x))}{(\cos^2(x))^2}
\]

\[
 = \frac{\cos^2(x) (\cos(x)) - \sin(x) (-2 \cos(x) \cdot (-\sin(x)))}{\cos^4(x)}
\]

\[
 = \frac{\cos^3(x) + 2 \sin^2(x) \cos(x)}{\cos^4(x)}
\]

\[
 = \frac{1}{\cos(x)} + 2 \frac{\sin^2(x)}{\cos^3(x)}.
\]

The easy way is to notice that

\[
 g(x) = \frac{\sin(x)}{\cos^2(x)} = \frac{1}{\cos(x)} \frac{\sin(x)}{\cos(x)} = \sec(x) \tan(x).
\]

Therefore, by the Product Rule,

\[
 g'(x) = (\sec(x) \tan(x)) \tan(x) + \sec(x) (\sec^2(x)) = \sec(x) (\tan^2(x) + \sec^2(x)).
\]

To see that the two answers we got are the same, remember that \( \tan^2(x) + 1 = \sec^2(x) \), so the second expression for \( g'(x) \) becomes

\[
 \sec(x) (\tan^2(x) + \tan^2(x) + 1) = \sec(x) (1 + 2 \tan^2(x)) = \frac{1}{\cos(x)} + 2 \frac{\sin^2(x)}{\cos^3(x)}.
\]

2. Let \( f(x) = 3^x \ln(x) \). What is \( f'(0) \)?

**Answer:** Unless you remember the derivative of \( 3^x \) and use the Chain Rule, the best way to solve this is to use logarithmic differentiation. Taking the natural log of both sides, we have

\[
 \ln(f(x)) = \ln(3^x \ln(x)) = x \ln(x) \ln(3).
\]

Differentiating both sides yields

\[
 \frac{1}{f(x)} \cdot f'(x) = \ln(3) \left( 1 \cdot \ln(x) + x \cdot \frac{1}{x} \right)
\]

\[
 \frac{f'(x)}{f(x)} = \ln(3) (\ln(x) + 1).
\]
Therefore,
\[ f'(x) = f(x) \ln(3)(\ln(x) + 1) \]
and, substituting back in for \( f(x) \) gives us the derivative of \( f \):
\[ f'(x) = 3^{x \ln(x)} \ln(3)(\ln(x) + 1). \]

Now, when we try to evaluate at \( x = 0 \), we run into a problem: \( \ln(0) \) is undefined! Therefore, \( f'(0) \) is undefined as well (in fact, we could have concluded this right at the beginning, since 0 is not in the domain of \( f \)).

3. The curve determined by the equation \( y^2 = x^2(x + 1) \), pictured below, is called the Tschirnhausen cubic. At what points does this curve have a horizontal tangent line?

\[ \begin{align*}
\text{Answer: Differentiating implicitly, we have} & \\
2y \cdot y' &= 2x(x + 1) + x^2(1) \\
\text{Thus,} & \\
y' &= \frac{2x^2 + 2x + x^2}{2y} = \frac{3x^2 + 2x}{2y}.
\end{align*} \]

The curve will have a horizontal tangent line only when the above is equal to zero. Clearly, the fraction can be zero only when the numerator is equal to zero:
\[ 0 = 3x^2 + 2x = x(3x + 2). \]

Hence, the numerator is equal to zero when \( x = 0 \) or \( x = -2/3 \). However (and this is the slightly tricky bit), when \( x \) is equal to zero, so is \( y \): in this case, both numerator and denominator are zero, so the expression isn’t necessarily zero. In fact, it’s clear from the graph above that the tangent line is not horizontal at the origin, so the graph has a horizontal tangent line only when \( x = -2/3 \). Using the original equation of the curve to solve for the \( y \)-coordinates corresponding to \( x = -2/3 \), we see that
\[ y^2 = (-2/3)^2(-2/3 + 1) = \frac{4}{9} \cdot \frac{1}{3} = \frac{4}{27}. \]

Therefore, \( y = \pm \sqrt{\frac{4}{27}} = \pm \frac{2}{3\sqrt{3}} \).

We conclude, then, that the Tschirnhausen cubic has a horizontal tangent line at the points
\[ \left( -\frac{2}{3}, -\frac{2}{3\sqrt{3}} \right) \text{ and } \left( -\frac{2}{3}, \frac{2}{3\sqrt{3}} \right). \]
4. The volume of a cube increases at a rate of 1 cm³ per minute. How fast is the surface area of the cube increasing when the length of an edge is 3 cm?

**Answer:** First, let’s collect the facts that we know. Letting $\ell(t)$ be the length of an edge at time $t$, $V(t)$ be the volume, $A(t)$ be the surface area, and $t_0$ be the time when the edge length is 3, we have

- $V(t) = \ell(t)^3$
- $A(t) = 6\ell(t)^2$ (remember, a cube has six faces)
- $V'(t) = 1$
- $\ell(t_0) = 3$

Our goal is to determine $A'(t_0)$. Notice that, using the above and the Chain Rule,

$$A'(t) = 6 \cdot 2\ell(t)\ell'(t) = 12\ell(t)\ell'(t).$$

Therefore, at the time of interest,

$$A'(t_0) = 12\ell(t_0)\ell'(t_0) = 12(3)\ell'(t_0) = 36\ell'(t_0).$$

If we can determine $\ell'(t_0)$ we’ll be able to substitute it into the above expression and determine $A'(t_0)$. We will be able to solve for $\ell'(t_0)$ by differentiating $V(t)$ and evaluating at $t = t_0$. First, we differentiate:

$$V'(t) = 3\ell(t)^2\ell'(t).$$

Since $V'(t) = 1$ for all $t$, we know

$$1 = 3\ell(t)^2\ell'(t),$$

so

$$\ell'(t) = \frac{1}{3\ell(t)^2}.$$}

Now we can just evaluate at $t = t_0$:

$$\ell'(t_0) = \frac{1}{3(3)^2} = \frac{1}{27}.$$}

Finally, then the expression for $A'(t_0)$ determined above yields

$$A'(t_0) = 36\ell'(t_0) = 36 \cdot \frac{1}{27} = \frac{4}{3}.$$}

Therefore, at the moment when the edges of the cube have length 3 cm, the surface area is growing at a rate of $\frac{4}{3}$ cm²/min.

5. Suppose that you are interested in computing the area of the right triangle pictured below and you determine that the right triangle is isosceles (this is easy to do using a compass). If you are able to measure the length of one of the short sides with a maximum error of 1%, how accurately can you compute the area of this right triangle?

![Triangle](image)
**Answer:** Recall, first, that the area of a triangle is

\[ A = \frac{1}{2}bh. \]

In the case of a right triangle like this one, \( b \) and \( h \) are just the lengths of the two short sides. Moreover, when the right triangle is isosceles, the two shorter sides are the same length; i.e. \( b = h \). Therefore

\[ A = \frac{1}{2}h \cdot h = \frac{1}{2}h^2. \]

Now, we intend to approximate the error in measuring the area using differentials. First, note that the differential of the area is

\[ dA = \frac{1}{2} \cdot 2hdh = hdh. \]

Also, we can measure the length of one of the shorter sides (which is to say, \( h \)) with a maximum error of 1%, which means that

\[ \frac{dh}{h} \leq 0.01. \]

The maximum relative error in measuring the area, then, is approximately

\[ \frac{dA}{A} = \frac{hdh}{\frac{1}{2}h^2} = 2\frac{dh}{h} \leq 2(0.01) = 0.02. \]

Therefore, the approximate maximum percentage error in measuring the area of the triangle is 2%.