Math 2250 Written HW #8 Solutions

1. The folium of Descartes, pictured below, is determined by the equation

\[ x^3 + y^3 - 9xy = 0. \]

(a) Determine the slopes of the tangent lines to the folium at the points (4, 2) and (2, 4).

**Answer:** We intend to determine \( y' \) at these points using implicit differentiation, so differentiate both sides of \( x^3 + y^3 - 9xy = 0 \) with respect to \( x \):

\[
\frac{d}{dx}(x^3 + y^3 - 9xy) = \frac{d}{dx}(0)
\]

\[
3x^2 + 3y^2y' - 9(1 \cdot y + xy') = 0
\]

\[
3x^2 + 3y^2y' - 9y - 9xy' = 0
\]

\[
3x^2 - 9y + y'(3y^2 - 9x) = 0.
\]

Therefore, we have

\[ y'(3y^2 - 9x) = 9y - 3x^2 \]

or, equivalently,

\[ y' = \frac{9y - 3x^2}{3y^2 - 9x}. \]

Hence, the slope of the tangent line at (4, 2) is

\[ y' = \frac{9(2) - 3(4)^2}{3(2)^2 - 9(4)} = \frac{18 - 48}{12 - 36} = \frac{-30}{-24} = \frac{5}{4}. \]
Likewise, the slope of the tangent line at \((2, 4)\) is
\[
y' = \frac{9(4) - 3(2)^2}{3(4)^2 - 9(2)} = \frac{36 - 12}{48 - 18} = \frac{24}{36} = \frac{4}{5}.
\]

(b) At what point other than the origin does the folium have a horizontal tangent line?  
**Answer:** The folium will have a horizontal tangent line when \(y' = 0\). Since we know \(y' = \frac{9y - 3x^2}{3y^2 - 9x}\), this will occur exactly when
\[
9y - 3x^2 = 0.
\]
Obviously, there are infinitely many such points, but we only care about the ones that are actually on the folium, which is to say, those that also satisfy the equation \(x^3 + y^3 - 9xy = 0\). If a point \((x, y)\) satisfies both of these equations then, from the first, we know that
\[
y = \frac{x^2}{3}.
\]
Plugging this in to the second, then, we see that
\[
x^3 + \left(\frac{x^2}{3}\right)^3 - 9x \left(\frac{x^2}{3}\right) = 0
\]
or, equivalently,
\[
x^3 + \frac{x^6}{27} - 3x^3 = 0.
\]
Combining and factoring, this means
\[
x^3 \left(\frac{x^3}{27} - 2\right) = 0,
\]
so either \(x = 0\) or \(\frac{x^3}{27} = 2\), meaning \(x = \sqrt[3]{54} = 3\sqrt[3]{2}\).  
\(x = 0\) just gives the origin, so we’re interested in the point where \(x = 3\sqrt[3]{2}\). The corresponding \(y\)-coordinate is
\[
y = \frac{(3\sqrt[3]{2})^2}{3} = \frac{9\sqrt[3]{4}}{3} = 3\sqrt[3]{4}.
\]
Therefore, the point other than the origin where the folium has a horizontal tangent line is \((3\sqrt[3]{2}, 3\sqrt[3]{4})\).

(c) At what point other than the origin does the folium have a vertical tangent line?  
**Answer:** Since we know \(y' = \frac{9y - 3x^2}{3y^2 - 9x}\), the curve will have a vertical tangent line when the denominator is zero, meaning
\[
3y^2 - 9x = 0.
\]
Again, we only care about such points that are also on the folium, meaning they also satisfy the equation \(x^3 + y^3 - 9xy = 0\).
But notice that we arrive at these equations from the equations in (b) above simply by interchanging \(x\) and \(y\). Therefore, the answer to this part will just be the answer to (b) with the \(x\)- and \(y\)-coordinates interchanged. In other words, the folium has a vertical tangent at the point \((3\sqrt{4}, 3\sqrt{2})\).

2. Let \(f(x) = x^{\cos x}\). What is \(f'(x)\)?

**Answer:** I will use logarithmic differentiation. First, take the natural log of both sides:

\[
\ln(f(x)) = \ln(x^{\cos x}),
\]

so

\[
\ln(f(x)) = \cos x \ln x.
\]

Now, differentiate both sides with respect to \(x\):

\[
\frac{d}{dx}(\ln(f(x))) = \frac{d}{dx}(\cos x \ln x)
\]

\[
\frac{1}{f(x)} f'(x) = -\sin x \ln x + \cos x \frac{1}{x}
\]

\[
\frac{f'(x)}{f(x)} = \frac{\cos x}{x} - \sin x \ln x.
\]

Therefore,

\[
f'(x) = f(x) \left( \frac{\cos x}{x} - \sin x \ln x \right).
\]

Substituting in \(f(x) = x^{\cos x}\), we see that

\[
f'(x) = x^{\cos x} \left( \frac{\cos x}{x} - \sin x \ln x \right).
\]

3. The length \(\ell\) of a rectangle is decreasing at a rate of 5 cm/sec while the width \(w\) is increasing at a rate of 3 cm/sec. At the moment when \(\ell = 11\) cm and \(w = 9\) cm, determine the following rates of change:

(a) The rate of change of the area (in cm\(^2\)/sec).

**Answer:** First, let’s get a handle on what we know. We know that the area is given by

\[
A(t) = \ell(t)w(t).
\]

Also, we know \(\ell'(t) = -5\), \(w'(t) = 3\), \(\ell(t_0) = 11\), and \(w(t_0) = 9\).

In this part, we’re trying to determine \(A'(t_0)\), so we just need to differentiate our expression for \(A(t)\) and evaluate at \(t = t_0\). Differentiating \(A(t) = \ell(t)w(t)\) yields

\[
A'(t) = \ell'(t)w(t) + \ell(t)w'(t).
\]

Therefore,

\[
A'(t_0) = \ell'(t_0)w(t_0) + \ell(t_0)w'(t_0) = (-5)(9) + (11)(3) = -45 + 33 = -12.
\]

Hence, we can see that the area is decreasing at a rate of 12 cm\(^2\)/sec at this moment.
(b) The rate of change of the perimeter (in cm/sec).

**Answer:** Now, we’re interested in the perimeter of the rectangle, which is given by

\[ P(t) = 2\ell(t) + 2w(t). \]

Hence,

\[ P'(t) = 2\ell'(t) + 2w'(t), \]

so we have

\[ P'(t_0) = 2\ell'(t_0) + 2w'(t_0) = 2(-5) + 2(3) = -10 + 6 = -4. \]

At this moment, then, the perimeter is decreasing at a rate of 4 cm/sec.

(c) The rate of change of the diagonals (in cm/sec).

**Answer:** From the Pythagorean theorem, if \( g \) is the length of a diagonal,

\[ g(t)^2 = \ell(t)^2 + w(t)^2. \]

We want to determine \( g'(t_0) \), so let’s differentiate the above expression:

\[ 2g(t)g'(t) = 2\ell(t)\ell'(t) + 2w(t)w'(t), \]

so we have

\[ g'(t) = \frac{\ell(t)\ell'(t) + w(t)w'(t)}{g(t)}. \]

Now, at time \( t = t_0 \), we know that

\[ g(t_0)^2 = \ell(t_0)^2 + w(t_0)^2 = 11^2 + 9^2 = 121 + 81 = 202, \]

so \( g(t_0) = \sqrt{202} \).

Therefore,

\[ g'(t_0) = \frac{(11)(-5) + (9)(3)}{\sqrt{202}} = \frac{-55 + 27}{\sqrt{202}} = \frac{-28}{\sqrt{202}}, \]

hence the diagonals are decreasing in length at \( \frac{28}{\sqrt{202}} \) cm/sec.