Math 2250 Written HW #11 Solutions

1. Show that the function \( f(x) = x^4 + 3x + 1 \) has exactly one zero in the interval \([-2, -1]\). (Hint: to show that \( f \) has at least one zero, use the Intermediate Value Theorem as in Section 2.5)

**Answer:** We attack this in two parts: (i) show that \( f(x) \) has at least one zero; and (ii) show that \( f(x) \) has at most one zero.

For (i), as suggested in the hint, the goal is to use the Intermediate Value Theorem. The IVT is surely applicable since \( f(x) \) is a polynomial and, hence, continuous everywhere. Now,

\[
\begin{align*}
  f(-2) &= (-2)^4 + 3(-2) + 1 = 16 - 6 + 1 = 11 > 0 \\
  f(-1) &= (-1)^4 + 3(-1) + 1 = 1 - 3 + 1 = -2 < 0.
\end{align*}
\]

Therefore, by the Intermediate Value Theorem there exists some number \( c \) between \(-2\) and \(-1\) so that \( f(c) = 0 \), and so we see that \( f(x) \) has at least one zero in the interval \([-2, -1]\).

For (ii), the strategy is to use the Mean Value Theorem. First, notice that \( f'(x) = 4x^3 + 3 \). Since \( 4x^3 + 3 \) is negative for all \( x \) in \([-2, -1]\), we see that \( f'(x) < 0 \) for all \( x \) in \([-2, -1]\).

On the other hand, if there were more than one zero of \( f(x) \) in this interval, there would be at least two, and so we could pick two, say \( a \) and \( b \) with \( a < b \). Then the Mean Value Theorem would imply that there exists \( c \) in \((a, b)\) so that

\[
  f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{0 - 0}{b - a} = 0.
\]

But we just saw that this is impossible since \( f'(x) < 0 \) for all \( x \) in \([-2, -1]\).

Therefore, we can conclude that there is at most one zero of \( f(x) \) in \([-2, -1]\). Combined with (i), then, this implies that there is exactly one zero of \( f(x) \) in \([-2, -1]\).

2. Suppose the acceleration of an oscillating particle is given by

\[
a(t) = -4 \sin(2t)
\]

at that the particle’s position at time \( t = 0 \) is \(-3\) and its velocity at time \( t = 0 \) is \( 2 \). Find the particle’s position as a function of \( t \).

**Answer:** Since \( a(t) = v'(t) \), if we can find some function \( g(t) \) so that \( g'(t) = a(t) = v'(t) \), then we’ll know that \( v(t) = g(t) + C \) for some constant \( C \), which we can then solve for using \( v(0) = 2 \).

To find such a \( g(t) \), notice \( \cos(2t) \) is the sort of function that has a derivative more or less like \(-4 \sin(2t)\). Specifically,

\[
\frac{d}{dt}(\cos(2t)) = -\sin(2t) \cdot 2 = -2 \sin(2t).
\]

Therefore, to get a function that has \( a(t) \) as a derivative, we should multiply \( \cos(2t) \) by \( 2 \):

\[
\frac{d}{dt}(2 \cos(2t)) = -2 \sin(2t) \cdot 2 = -4 \sin(2t).
\]
Thus,

\[ v(t) = 2 \cos(2t) + C \]

for some constant \( C \) which we now solve for using \( v(0) = 2 \):

\[
\begin{align*}
    v(0) &= 2 \cos(2 \cdot 0) + C \\
    2 &= 2 \cos(0) + C \\
    2 &= 2 + C
\end{align*}
\]

so \( C = 0 \), and we have that \( v(t) = 2 \cos(2t) \).

Next, we use the same strategy to determine the position function \( s(t) \), using the fact that \( s'(t) = v(t) \). So now we seek a function with \( 2 \cos(2t) \) as its derivative. This is a bit easier:

\[
\frac{d}{dt}(\sin(2t)) = \cos(2t) \cdot 2 = 2 \cos(2t).
\]

Hence,

\[ s(t) = \sin(2t) + D \]

for some constant \( D \) which we can determine using \( s(0) = -3 \):

\[
\begin{align*}
    s(0) &= \sin(2 \cdot 0) + D \\
    -3 &= 0 + D \\
    -3 &= D.
\end{align*}
\]

From all this, then, we see that the position of the particle at time \( t \) is given by the function

\[ s(t) = \sin(2t) - 3. \]

3. A trucker handed in a ticket at a toll booth showing that in 2 hours he had covered 159 miles on a toll road with a speed limit of 65 mph. He was immediately cited for speeding and, when he asked for an explanation, the only response was “Mean Value Theorem, son.” Explain.

**Answer:** Let \( s(t) \) denote the position of the trucker at time \( t \) (starting at \( t = 0 \) when he enters the toll road), and let \( v(t) = s'(t) \) be his velocity. Then, by the Mean Value Theorem, there exists some time \( t_0 \) so that

\[
v(t_0) = s'(t_0) = \frac{s(2) - s(0)}{2 - 0} = \frac{159}{2} = 79.5.
\]

That means there was some time when the trucker was driving exactly 79.5 mph, which is clearly well above the 65 mph speed limit; hence the speeding ticket.