Math 215 HW #7 Solutions

1. Problem 3.3.8. If $P$ is the projection matrix onto a $k$-dimensional subspace $S$ of the whole space $\mathbb{R}^n$, what is the column space of $P$ and what is its rank?

**Answer:** The column space of $P$ is $S$. To see this, notice that, if $\vec{x} \in \mathbb{R}^n$, then $P\vec{x} \in S$ since $P$ projects $\vec{x}$ to $S$. Therefore, $\text{col}(P) \subset S$. On the other hand, if $\vec{b} \in S$, then $P\vec{b} = \vec{b}$, so $S \subset \text{col}(P)$. Since containment goes both ways, we see that $\text{col}(P) = S$.

Therefore, since the rank of $P$ is equal to the dimension of $\text{col}(P) = S$ and since $S$ is $k$-dimensional, we see that the rank of $P$ is $k$.

2. Problem 3.3.12. If $V$ is the subspace spanned by $(1,1,0,1)$ and $(0,0,1,0)$, find

(a) a basis for the orthogonal complement $V^\perp$.

**Answer:** Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$  

By construction, the row space of $A$ is equal to $V$. Therefore, since the nullspace of any matrix is the orthogonal complement of the row space, it must be the case that $V^\perp = \text{nul}(A)$. The matrix $A$ is already in reduced echelon form, so we can see that the homogeneous equation $A\vec{x} = \vec{0}$ is equivalent to

$$x_1 = -x_2 - x_4, \\ x_3 = 0.$$  

Therefore, the solutions of the homogeneous equation are of the form

$$x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$  

so the following is a basis for $\text{nul}(A) = V^\perp$:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$  

(b) the projection matrix $P$ onto $V$.

**Answer:** From part (a), we have that $V$ is the row space of $A$ or, equivalently, $V$ is the column space of

$$B = A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$
Therefore, the projection matrix $P$ onto $V = \text{col}(B)$ is

$$P = B(B^TB)^{-1}B^T = A^T(AA^T)^{-1}A.$$ 

Now,

$$B^TB = AA^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix},$$

so

$$(AA^T)^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}. $$

Therefore,

$$P = A^T(AA^T)^{-1}A$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

(c) the vector in $V$ closest to the vector $\vec{b} = (0, 1, 0, -1)$ in $V^\perp$.

**Answer:** The closest vector to $\vec{b}$ in $V$ will necessarily be the projection of $\vec{b}$ onto $V$. Since $\vec{b}$ is perpendicular to $V$, we know this will be the zero vector. We can also double-check this since the projection of $\vec{b}$ onto $V$ is

$$P\vec{b} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ 

3. Problem 3.3.22. Find the best line $C + Dt$ to fit $b = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

**Answer:** If the above data points actually lay on a straight line $C + Dt$, we would have

$$\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$
Call the matrix \( A \) and the vector on the right-hand side \( \vec{b} \). Of course this system is inconsistent, but we want to find \( \hat{x} = \begin{bmatrix} C \\ D \end{bmatrix} \) such that \( A\hat{x} \) is as close as possible to \( \vec{b} \). As we’ve seen, the correct choice of \( \hat{x} \) is given by

\[
\hat{x} = (A^TA)^{-1}A^T\vec{b}.
\]

To compute this, first note that

\[
A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}.
\]

Therefore,

\[
(A^TA)^{-1} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{10} \end{bmatrix}
\]

and so

\[
\hat{x} = (A^TA)^{-1}A^T\vec{b}
\]

\[
= \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} \frac{1}{5} & 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} -8 \\ 5 \end{bmatrix}
\]

\[
= \begin{bmatrix} \frac{1}{4} \end{bmatrix}
\]

Therefore, the best-fit line for the data is

\[1 - \frac{4}{5}t.\]

Here are the data points and the best-fit line on the same graph:
4. Problem 3.3.24. Find the best straight-line fit to the following measurements, and sketch
your solution:
\[ y = 2 \text{ at } t = -1, \ y = 0 \text{ at } t = 0, \]
\[ y = -3 \text{ at } t = 1, \ y = -5 \text{ at } t = 2. \]

**Answer:** As in Problem 3, if the data actually lay on a straight line \( y = C + Dt \), we would have
\[
\begin{bmatrix}
1 & -1 \\
1 & 0 \\
1 & 1 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
C \\
D
\end{bmatrix}
=
\begin{bmatrix}
2 \\
0 \\
-3 \\
-5
\end{bmatrix}.
\]

Again, this system is not solvable, but, if \( A \) is the matrix and \( \vec{b} \) is the vector on the right-hand side, then we want to find \( \hat{x} \) such that \( A\hat{x} \) is as close as possible to \( \vec{b} \). This will happen when
\[
\hat{x} = (A^T A)^{-1} A^T \vec{b}.
\]

Now,
\[
A^T A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
-1 & 0 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 0 \\
1 & 1 \\
1 & 2
\end{bmatrix}
= \begin{bmatrix}
4 & 2 \\
2 & 6
\end{bmatrix}.
\]

To find \((A^T A)^{-1}\), we want to perform row operations on the augmented matrix
\[
\begin{bmatrix}
4 & 2 & 1 & 0 \\
2 & 6 & 0 & 1
\end{bmatrix}
\]
so that the \( 2 \times 2 \) identity matrix appears on the left. To that end, scale the first row by \( \frac{1}{4} \) and subtract 2 times the result from row 2:
\[
\begin{bmatrix}
1 & 1/2 & 1/4 & 0 \\
0 & 5 & -1/2 & 1
\end{bmatrix}.
\]

Now, scale row 2 by \( \frac{1}{5} \) and subtract half the result from row 1:
\[
\begin{bmatrix}
1 & 0 & 3/10 & -1/10 \\
0 & 1 & -1/10 & 1/5
\end{bmatrix}.
\]

Therefore,
\[
(A^T A)^{-1} = \begin{bmatrix}
3/10 & -1/10 \\
-1/10 & 1/5
\end{bmatrix}
\]
and so

\[ \hat{x} = (A^T A)^{-1} A^T \vec{b} \]

\[ = \begin{bmatrix} 3/10 & -1/10 \\ -1/10 & 1/5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix} \]

\[ = \begin{bmatrix} 3/10 & -1/10 \\ -1/10 & 1/5 \end{bmatrix} \begin{bmatrix} -6 \\ -15 \end{bmatrix} \]

\[ = \begin{bmatrix} -3/10 \\ -12/5 \end{bmatrix}. \]

Therefore, the best-fit line for the data is

\[ y = -\frac{3}{10} - \frac{12}{5} t. \]

Here’s a plot of both the data and the best-fit line:

5. Problem 3.3.25. Suppose that instead of a straight line, we fit the data in Problem 24 (i.e. #3 above) by a parabola \( y = C + Dt + Et^2 \). In the inconsistent system \( A\vec{x} = \vec{b} \) that comes from the four measurements, what are the coefficient matrix \( A \), the unknown vector \( \vec{x} \), and the data vector \( \vec{b} \)? For extra credit, actually determine the best-fit parabola.

\textbf{Answer:} Since the data hasn’t changed, the data vector \( \vec{b} \) will be the same as in the previous problem. If the data were to lie on a parabola \( C + Dt + Et^2 \), then we would have that

\[ \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix}, \]

so \( A \) is the matrix above and \( \vec{x} \) is the vector next to \( A \) on the left-hand side.

To actually determine the best-fit parabola, we just need to find \( \hat{x} \) such that \( A\hat{x} \) is as close as possible to \( \vec{b} \). This will be the vector

\[ \hat{x} = (A^T A)^{-1} A^T \vec{b}. \]
Now,
\[ A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix} \]

To find \((A^T A)^{-1}\), we want to use row operations to convert the left-hand side of this augmented matrix to \(I\):
\[
\begin{bmatrix} 4 & 2 & 6 & 1 & 0 & 0 \\ 2 & 6 & 8 & 0 & 1 & 0 \\ 6 & 8 & 18 & 0 & 0 & 1 \end{bmatrix}.
\]

First, scale row 1 by \(\frac{1}{4}\) and subtract twice the result from row 2 and six times the result from row 3:
\[
\begin{bmatrix} 1 & 1/2 & 3/2 & 1/4 & 0 & 0 \\ 0 & 5 & 5 & -1/2 & 1 & 0 \\ 0 & 5 & 9 & -3/2 & 0 & 1 \end{bmatrix}.
\]

Next, subtract row 2 from row 3, scale row 2 by \(\frac{1}{5}\) and subtract half the result from row 1:
\[
\begin{bmatrix} 1 & 0 & 1 & 3/10 & -1/10 & 0 \\ 0 & 1 & 1 & -1/10 & 1/5 & 0 \\ 0 & 0 & 4 & -1 & -1 & 1 \end{bmatrix}.
\]

Finally, scale row 3 by \(\frac{1}{4}\) and subtract the result from rows 1 and 2:
\[
\begin{bmatrix} 1 & 0 & 0 & 11/20 & 3/20 & -1/4 \\ 0 & 1 & 0 & 3/20 & 9/20 & -1/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & 1/4 \end{bmatrix}.
\]

Therefore,
\[(A^T A)^{-1} = \begin{bmatrix} 11/20 & 3/20 & -1/4 \\ 3/20 & 9/20 & -1/4 \\ -1/4 & -1/4 & 1/4 \end{bmatrix}\]

and so
\[
\hat{x} = (A^T A)^{-1} A^T \vec{b}
\]
\[
= \begin{bmatrix} 11/20 & 3/20 & -1/4 \\ 3/20 & 9/20 & -1/4 \\ -1/4 & -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}
\]
\[
= \begin{bmatrix} 11/20 & 3/20 & -1/4 \\ 3/20 & 9/20 & -1/4 \\ -1/4 & -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} -6 \\ -15 \\ -21 \end{bmatrix}
\]
\[
= \begin{bmatrix} -3/10 \\ -12/5 \\ -1/4 \end{bmatrix}.
\]
Thus, the best-fit parabola is
\[ y = -\frac{3}{10} - \frac{12}{5} t + 0t^2 = -\frac{3}{10} - \frac{12}{5} t, \]
which is the same as the best-fit line!

6. Problem 3.4.4. If \(Q_1\) and \(Q_2\) are orthogonal matrices, so that \(Q_1^T Q_2 = I\), show that \(Q_1 Q_2\) is also orthogonal. If \(Q_1\) is rotation through \(\theta\) and \(Q_2\) is rotation through \(\phi\), what is \(Q_1 Q_2\)? Can you find the trigonometric identities for \(\sin(\theta + \phi)\) and \(\cos(\theta + \phi)\) in the matrix multiplication \(Q_1 Q_2\)?

**Answer:** Note that

\[
(Q_1 Q_2)^T (Q_1 Q_2) = Q_2^T Q_1^T Q_1 Q_2 = Q_2^T Q_2 = I,
\]
since both \(Q_1\) and \(Q_2\) are orthogonal matrices. Therefore, the columns of \(Q_1 Q_2\) are orthonormal. Moreover, since both \(Q_1\) and \(Q_2\) are square and must be the same size for \(Q_1 Q_2\) to make sense, it must be the case that \(Q_1 Q_2\) is square. Therefore, since \(Q_1 Q_2\) is square and has orthonormal columns, it is an orthogonal matrix.

If \(Q_1\) is rotation through angle \(\theta\), then, as we’ve seen,

\[
Q_1 = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}.
\]

Likewise, if \(Q_2\) is rotation through angle \(\phi\), then

\[
Q_2 = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix}.
\]

With these choices of \(Q_1\) and \(Q_2\), if \(\vec{x}\) is any vector in the plane \(\mathbb{R}^2\), we see that

\[
Q_1 Q_2 \vec{x} = Q_1 (Q_2 \vec{x}),
\]
meaning that \(\vec{x}\) is first rotated by an angle \(\phi\), then the result is rotated by an angle \(\theta\). Of course, this is the same as rotating \(\vec{x}\) by an angle \(\theta + \phi\), so \(Q_1 Q_2\) is precisely the matrix of the transformation which rotates the plane through an angle of \(\theta + \phi\). On the one hand, we know that

\[
Q_1 Q_2 = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} = \begin{bmatrix} \cos \theta \cos \phi - \sin \theta \sin \phi & \cos \theta \sin \phi + \sin \theta \cos \phi \\ -\sin \theta \cos \phi - \cos \theta \sin \phi & -\sin \theta \sin \phi + \cos \theta \cos \phi \end{bmatrix}.
\]

On the other hand, the matrix which rotates the plane through an angle of \(\theta + \phi\) is precisely

\[
\begin{bmatrix} \cos(\theta + \phi) & \sin(\theta + \phi) \\ -\sin(\theta + \phi) & \cos(\theta + \phi) \end{bmatrix}.
\]

Hence, it must be the case that

\[
\begin{bmatrix} \cos(\theta + \phi) & \sin(\theta + \phi) \\ -\sin(\theta + \phi) & \cos(\theta + \phi) \end{bmatrix} = \begin{bmatrix} \cos \theta \cos \phi - \sin \theta \sin \phi & \cos \theta \sin \phi + \sin \theta \cos \phi \\ -\sin \theta \cos \phi - \cos \theta \sin \phi & -\sin \theta \sin \phi + \cos \theta \cos \phi \end{bmatrix}.
\]

This implies the following trigonometric identities:

\[
\begin{align*}
\cos(\theta + \phi) &= \cos \theta \cos \phi - \sin \theta \sin \phi \\
\sin(\theta + \phi) &= \cos \theta \sin \phi + \sin \theta \cos \phi
\end{align*}
\]
7. Problem 3.4.6. Find a third column so that the matrix

\[
Q = \begin{bmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\
\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \\
\frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}}
\end{bmatrix}
\]

is orthogonal. It must be a unit vector that is orthogonal to the other columns; how much freedom does this leave? Verify that the rows automatically become orthonormal at the same time.

**Answer:** Let \( \vec{q}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \) and \( \vec{q}_2 = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ -3/\sqrt{14} \end{bmatrix} \). If \( \vec{q}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \) such that \( \|\vec{q}_3\| = 1 \), \( \langle \vec{q}_1, \vec{q}_3 \rangle = 0 \) and \( \langle \vec{q}_2, \vec{q}_3 \rangle = 0 \) then we have that

\[
1 = \|\vec{q}_3\|^2 = \langle \vec{q}_3, \vec{q}_3 \rangle = a^2 + b^2 + c^2 \\
0 = \langle \vec{q}_1, \vec{q}_3 \rangle = a/\sqrt{3} + b/\sqrt{3} + c/\sqrt{3} \\
0 = \langle \vec{q}_2, \vec{q}_3 \rangle = a/\sqrt{14} + 2b/\sqrt{14} - 3c/\sqrt{14}.
\]

Multiplying the second line by \( \sqrt{3} \) and the third line by \( \sqrt{14} \), we get the equivalent system

\[
1 = a^2 + b^2 + c^2 \\
0 = a + b + c \\
0 = a + 2b - 3c
\]

From the second line we have that \( b = -a - c \) and so, from the third line,

\[
a = -2b + 3c = -2(-a - c) + 3c = 2a + 5c.
\]

Thus \( a = -5c \), meaning that \( b = -a - c = -(5c) - c = 4c \). Therefore

\[
1 = a^2 + b^2 + c^2 = (-5c)^2 + (4c)^2 + c^2 = 42c^2,
\]

meaning that \( c = \pm 1/\sqrt{42} \). Thus, we see that

\[
\vec{q}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -5c \\ 4c \\ c \end{bmatrix} = \pm \begin{bmatrix} -5/\sqrt{42} \\ 4/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix}.
\]

Therefore, there are two possible choices; one of them gives the following orthogonal matrix:

\[
Q = \begin{bmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} & -\frac{5}{\sqrt{42}} \\
\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & \frac{4}{\sqrt{42}} \\
\frac{1}{\sqrt{3}} & -3/\sqrt{14} & \frac{1}{\sqrt{42}}
\end{bmatrix}.
\]

It is straightforward to check that each row has length 1 and is perpendicular to the other rows.
8. Problem 3.4.12. What multiple of \( \vec{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) should be subtracted from \( \vec{a}_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \) to make the result orthogonal to \( \vec{a}_1 \)? Factor \( \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} \) into \( QR \) with orthonormal vectors in \( Q \).

**Answer:** Let’s do Gram-Schmidt on \( \{\vec{a}_1, \vec{a}_2\} \). First, we let
\[
\vec{v}_1 = \frac{\vec{a}_1}{\|\vec{a}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]
Next,
\[
\vec{w}_2 = \vec{a}_2 - (\langle \vec{v}_1, \vec{a}_2 \rangle) \vec{v}_1 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - 4/\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.
\]
By construction, \( \vec{w}_2 \) is orthogonal to \( \vec{a}_1 \), so we see that we needed to subtract 2 times \( \vec{a}_1 \) from \( \vec{a}_2 \) to get a vector perpendicular to \( \vec{a}_1 \).

Now, continuing with Gram-Schmidt, we get that
\[
\vec{v}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}.
\]
Therefore, if \( A = [\vec{a}_1 \ \vec{a}_2] \) and \( Q = [\vec{v}_1 \ \vec{v}_2] \), then
\[
A = QR
\]
where
\[
R = \begin{bmatrix} \langle \vec{a}_1, \vec{v}_1 \rangle & \langle \vec{a}_2, \vec{v}_1 \rangle \\ 0 & \langle \vec{a}_2, \vec{v}_2 \rangle \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix}.
\]

9. Problem 3.4.18. If \( A = QR \), find a simple formula for the projection matrix \( P \) onto the column space of \( A \).

**Answer:** If \( A = QR \), then
\[
A^T A = (QR)^T (QR) = R^T Q^T QR = R^T IR = R^T R,
\]
since \( Q \) is an orthogonal matrix (meaning \( Q^T Q = I \)). Thus, the projection matrix \( P \) onto the column space of \( A \) is given by
\[
P = A(A^T A)^{-1} A^T = QR(R^T R)^{-1}(QR)^T = QRR^{-1}(R^T)^{-1}R^T Q^T = QQ^T
\]
(provided, of course, that \( R \) is invertible).

10. Problem 3.4.32.

(a) Find a basis for the subspace \( S \) in \( \mathbb{R}^4 \) spanned by all solutions of
\[
x_1 + x_2 + x_3 - x_4 = 0.
\]
**Answer:** The solutions of the given equation are, equivalently, solutions of the matrix equation

\[
\begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0,
\]

so \( S \) is the nullspace of the \( 1 \times 4 \) matrix \( A = [1 \ 1 \ 1 \ -1] \). Since \( A \) is already in reduced echelon form, we can read off that the solutions to the above matrix equation are the vectors of the form

\[
x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.
\]

Therefore, a basis for \( \text{nul}(A) = S \) is given by

\[
\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.
\]

(b) Find a basis for the orthogonal complement \( S^\perp \).

**Answer:** Since \( S = \text{nul}(A) \), it must be the case that \( S^\perp \) is the row space of \( A \). Hence, the one row of \( A \) gives a basis for \( S^\perp \), meaning that the following is a basis for \( S^\perp \):

\[
\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}.
\]

(c) Find \( \bar{b}_1 \) in \( S \) and \( \bar{b}_2 \) in \( S^\perp \) so that \( \bar{b}_1 + \bar{b}_2 = \bar{b} = (1, 1, 1, 1) \).

**Answer:** For any \( \bar{b}_1 \in S \), we know that \( \bar{b}_1 \) is a linear combination of elements of the basis for \( S \) that we found in part (a). In other words,

\[
\bar{b}_1 = a \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

for some choice of \( a, b, c \in \mathbb{R} \). Also, if \( \bar{b}_2 \in S^\perp \), then \( \bar{b}_2 \) is a multiple of the basis vector for \( S^\perp \) we found in part (b). Thus,

\[
\bar{b}_2 = d \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}
\]
for some $d \in \mathbb{R}$. Therefore,

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

or, equivalently,

$$\begin{bmatrix} -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$  

To solve this matrix equation, we just do elimination on the augmented matrix

$$\begin{bmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}.$$  

Add row 1 to row 2:

$$\begin{bmatrix} -1 & -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}.$$  

Next, add row 2 to row 3:

$$\begin{bmatrix} -1 & -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}.$$  

Finally, subtract row 3 from row 4:

$$\begin{bmatrix} -1 & -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & -4 & -2 \end{bmatrix}.$$  

Therefore, $-4d = -2$, so $d = \frac{1}{2}$. Hence,

$$3 = c + 3d = c + \frac{3}{2},$$

so $c = \frac{3}{2}$. In turn,

$$2 = -b + c + 2d = -b + \frac{3}{2} + 1 = -b + \frac{5}{2},$$

meaning $b = \frac{1}{2}$. Finally,

$$1 = -a - b + c + d = -a - \frac{1}{2} + \frac{3}{2} + \frac{1}{2} = -a + \frac{3}{2},$$
so \( a = -\frac{1}{2} \).

Therefore,

\[
\vec{b}_1 = -\frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \\ 1/2 \\ 3/2 \end{bmatrix}
\]

and

\[
\vec{b}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}
\]

11. **(Bonus Problem)** Problem 3.4.24. Find the fourth Legendre polynomial. It is a cubic \( x^3 + ax^2 + bx + c \) that is orthogonal to \( 1, x, \) and \( x^2 - \frac{1}{3} \) over the interval \( -1 \leq x \leq 1 \).

**Answer:** We can find the fourth Legendre polynomial in the same style as Strang finds the third Legendre polynomial on p. 185:

\begin{equation}
\begin{aligned}
v_4 &= x^3 - \frac{\langle 1, x^3 \rangle}{\langle 1, 1 \rangle} - \frac{\langle x, x^3 \rangle}{\langle x, x \rangle} x - \frac{\langle x^2 - \frac{1}{3}, x^3 \rangle}{\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3} \rangle} \left( x^2 - \frac{1}{3} \right).
\end{aligned}
\end{equation}

Now, we just compute each of the inner products in turn:

\begin{align*}
\langle 1, x^3 \rangle &= \int_{-1}^{1} x^3 dx = 0 \\
\langle 1, 1 \rangle &= \int_{-1}^{1} 1 dx = 2 \\
\langle x, x^3 \rangle &= \int_{-1}^{1} x^4 dx = \frac{2}{5} \\
\langle x, x \rangle &= \int_{-1}^{1} x^2 dx = \frac{2}{3} \\
\langle x^2 - \frac{1}{3}, x^3 \rangle &= \int_{-1}^{1} \left( x^5 - \frac{x^3}{3} \right) dx = 0 \\
\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3} \rangle &= \int_{-1}^{1} \left( x^2 - \frac{1}{3} \right)^2 dx = \frac{8}{45}.
\end{align*}

Therefore, (1) becomes

\[v_4 = x^3 - 0 \cdot 1 - \frac{2/5}{2/3} x - 0 \cdot \left( x^2 - \frac{1}{3} \right) = x^3 - \frac{3}{5} x.\]

Therefore, the fourth Legendre polynomial is \( x^3 - \frac{3}{5} x \).