Math 215 HW #4 Solutions

1. Problem 2.1.6. Let $P$ be the plane in 3-space with equation $x + 2y + z = 6$. What is the equation of the plane $P_0$ through the origin parallel to $P$? Are $P$ and $P_0$ subspaces of $\mathbb{R}^3$?

Answer: For any real number $r$, the plane $x + 2y + z = r$ is parallel to $P$, since all such planes have a common normal vector $\mathbf{i} + 2\mathbf{j} + \mathbf{k} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. In particular, notice that the plane determined by the equation $x + 2y + z = 0$ is parallel to $P$ and passes through the origin (since $(x, y, z) = (0, 0, 0)$ is a solution of the above equation). Hence, this is the equation which determines the plane $P_0$.

Now, suppose $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in P_0$; i.e. the triples $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ both satisfy the equation $(\ast)$. Then

$$(x_1 + x_2) + 2(y_1 + y_2) + (z_1 + z_2) = (x_1 + 2y_1 + z_1) + (x_2 + 2y_2 + z_2) = 0 + 0 = 0,$$

so we have that

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} \in P_0.$$

Also, if $c \in \mathbb{R}$, then

$$cx_1 + 2(cy_1) + cz_1 = c(x_1 + 2y_1 + z_1) = c(0) = 0,$$

so

$$c \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix} \in P_0.$$

Therefore, $P_0$ is a subspace of $\mathbb{R}^3$.

On the other hand, $\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$ are both in $P$, but

$$\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

is not in $P$ since $6 + 2(3) + 0 = 12 \neq 6$.

Therefore, we see that $P$ is not a subspace of $\mathbb{R}^3$. 

2. Problem 2.1.12. The functions \( f(x) = x^2 \) and \( g(x) = 5x \) are “vectors” in the vector space \( \mathbb{F} \) of all real functions. The combination \( 3f(x) - 4g(x) \) is the function \( h(x) = \) ________. Which rule is broken if multiplying \( f(x) \) by \( c \) gives the function \( f(cx) \)?

**Answer:** The combination \( 3f(x) - 4g(x) \) is the function

\[
h(x) = 3x^2 - 20x.
\]

If we tried to define scalar multiplication as \( cf(x) = f(cx) \) we would run into problems. Note that

\[
f(5x) = (5x)^2 = 25x^2; \]

but

\[
f(2x) + f(3x) = (2x)^2 + (3x)^2 = 4x^2 + 9x^2 = 13x^2.
\]

Hence, this attempted definition of scalar multiplication would not satisfy rule 8 in the definition of a vector space.

3. Problem 2.1.18.

(a) The intersection of two planes through \((0,0,0)\) is probably a ________ but it could be a ________. It can’t be the zero vector \( \mathbb{Z}! \)

**Answer:** The intersection of two planes through the origin in \( \mathbb{R}^3 \) is probably a line, but it could be a plane (if the two planes coincide).

(b) The intersection of a plane through \((0,0,0)\) with a line through \((0,0,0)\) is probably a ________ but it could be a ________.

**Answer:** The intersection of a plane through the origin with a line through the origin in \( \mathbb{R}^3 \) is probably just the single point \((0,0,0)\), but it could be a whole line (if the line lies in the plane).

(c) If \( S \) and \( T \) are subspaces of \( \mathbb{R}^5 \), their intersection \( S \cap T \) (vectors in both subspaces) is a subspace of \( \mathbb{R}^5 \). *Check the requirements on \( x + y \) and \( cx \).*

**Answer:** To see that \( S \cap T \) is a subspace, suppose \( x, y \in S \cap T \) and that \( c \in \mathbb{R} \). Then, since \( x \) and \( y \) are both in \( S \) and since \( S \) is a subspace (meaning that it is closed under addition), we have that

\[
x + y \in S.
\]

 Likewise, since \( x \) and \( y \) are both elements of \( T \) and since \( T \) is a subspace, we have that \( x + y \in T \). Therefore, since \( x + y \) is in both \( S \) and \( T \), we have that

\[
x + y \in S \cap T.
\]

Likewise, since \( x \in S \) and \( S \) is a subspace (meaning that \( S \) is closed under scalar multiplication), we have that \( cx \in S \); similarly, \( cx \in T \). Therefore,

\[
  cx \in S \cap T.
\]

Since our choices of \( x, y, \) and \( c \) were completely arbitrary, we see that \( S \cap T \) is a subspace of \( \mathbb{R}^5 \).
Problem 2.1.22. For which right-hand sides (find a condition on $b_1, b_2, b_3$) are these systems solvable?

(a) \[
\begin{bmatrix}
1 & 4 & 2 \\
2 & 8 & 4 \\
-1 & -4 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3
\end{bmatrix}
= \begin{bmatrix}
b_1 \\ b_2 \\ b_3
\end{bmatrix}.
\]

(b) \[
\begin{bmatrix}
1 & 4 \\
2 & 9 \\
-1 & -4
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2
\end{bmatrix}
= \begin{bmatrix}
b_1 \\ b_2 \\ b_3
\end{bmatrix}.
\]

(a) **Answer:** Form the augmented matrix
\[
\begin{bmatrix}
1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3
\end{bmatrix}
.
\]
The goal is to use elimination to get this into reduced echelon form. Subtract twice row 1 from row 2 and add row 1 to row 3 to get:
\[
\begin{bmatrix}
1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + b_1
\end{bmatrix}.
\]
Hence, the given equation is solvable only if
\[
b_2 - 2b_1 = 0 \quad \text{and} \quad b_3 + b_1 = 0.
\]
In other words, the right-hand side of the equation must be a vector of the form
\[
\begin{bmatrix}
b_1 \\ 2b_1 \\ -b_1
\end{bmatrix}
= b_1 \begin{bmatrix}
1 \\ 2 \\ -1
\end{bmatrix}
\]
for any real number $b_1$. In other words, the column space of the given matrix is the line containing the vector $\begin{bmatrix}
1 \\ 2 \\ -1
\end{bmatrix}$.

(b) **Answer:** Form the augmented matrix
\[
\begin{bmatrix}
1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3
\end{bmatrix}.
\]
Then the goal is to get this into reduced echelon form. To do so, subtract twice row 1 from row 2 and add row 1 to row 3, yielding:
\[
\begin{bmatrix}
1 & 4 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_3 + b_1
\end{bmatrix}.
\]
The given equation is solvable only if
\[
b_3 + b_1 = 0,
\]
or, equivalently, if $b_3 = -b_1$. Hence, the possible right-hand sides are vectors of the form

$$
\begin{bmatrix}
  b_1 \\
  b_2 \\
  -b_1
\end{bmatrix}
= b_1 \begin{bmatrix}
  1 \\
  0 \\
  -1
\end{bmatrix}
+ b_2 \begin{bmatrix}
  0 \\
  1 \\
  0
\end{bmatrix}.
$$

In other words, the column space of the given matrix is the plane containing the vectors

$$
\begin{bmatrix}
  1 \\
  0 \\
  -1
\end{bmatrix}
\text{ and }
\begin{bmatrix}
  0 \\
  1 \\
  0
\end{bmatrix}.
$$

5. Problem 2.1.28. True or false (with a counterexample if false)?

(a) The vectors $b$ that are not in the column space $\text{C}(A)$ form a subspace.

**Answer:** False. Let

$$
A = \begin{bmatrix}
  1 & 0 \\
  0 & 0
\end{bmatrix}.
$$

Then, for any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, it’s clear that

$$
Ax = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.
$$

Hence, the column space of $A$ consists of all vectors of the form $\begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ for some real number $x_1$. Now, neither of the two vectors

$$
\begin{bmatrix}
  1 \\
  1
\end{bmatrix},
\begin{bmatrix}
  0 \\
  -1
\end{bmatrix}
$$

is in the column space of $A$, but their sum

$$
\begin{bmatrix}
  1 \\
  1
\end{bmatrix} + \begin{bmatrix}
  0 \\
  -1
\end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

is in $\text{C}(A)$. Therefore, the vectors that are not in $\text{C}(A)$ do are not closed under addition and so do not form a subspace.

(b) If $\text{C}(A)$ contains only the zero vector, then $A$ is the zero matrix.

**Answer:** True. The column space of $A$ consists of all linear combinations of the columns of $A$. In particular, each column of $A$ is an element of $\text{C}(A)$. Hence, if $\text{C}(A)$ contains only the zero vector, then each column of $A$ must be the zero vector, meaning that $A$ is the zero matrix.

(c) The column space of $2A$ equals the column space of $A$.

**Answer:** True. Suppose $b$ is in the column space of $A$. That means there exists some $x$ such that $Ax = b$. Then

$$
\left(2A\right)\left(\frac{1}{2}x\right) = Ax = b,
$$
so $b$ is in the column space of $2A$. Hence, the column space of $A$ is contained in the column space of $2A$.

On the other hand, if $c$ is in the column space of $2A$, then there exists $x$ such that $(2A)x = c$. But that means that

$$A(2x) = 2Ax = c,$$

so $c$ is also in the column space of $A$. Hence, the column space of $2A$ is contained in the column space of $A$.

Since we’ve shown containments both directions, it must be the case that the column space of $A$ and the column space of $2A$ are the same space.

(d) The column space of $A - I$ equals the column space of $A$.

**Answer:** False. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

Then the column space of $A$ consists of all linear combinations of the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which is to say all of $\mathbb{R}^2$. On the other hand,

$$A - I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so the column space of $A - I$ consists only of the zero matrix.

6. Problem 2.2.6. Describe the attainable right-hand sides $b$ (in the column space) for

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

by finding the constraints on $b$ that turn the third equation into $0 = 0$ (after elimination). What is the rank, and a particular solution?

**Answer:** Consider the augmented matrix

$$\begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 2 & 3 & b_3 \end{bmatrix}.$$ 

We can convert this to reduced echelon form by subtracting twice row 1 from row 3 and subtracting 3 times row 2 from row 3:

$$\begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & b_3 - 2b_1 - 3b_2 \end{bmatrix}.$$ 

In order for this system to be consistent, it must be the case that

$$b_3 - 2b_1 - 3b_2 = 0.$$
or, equivalently,
\[ b_3 = 2b_1 + 3b_2. \]

There are no constraints on \( b_1 \) and \( b_2 \), the possible right-hand sides of the equation are vectors of the form
\[
\begin{bmatrix}
  b_1 \\
  b_2 \\
  2b_1 + 3b_2
\end{bmatrix}
= b_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}
\]
for any real numbers \( b_1 \) and \( b_2 \). In other words, the column space of \( A \) is the plane containing the vectors \( \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \).

Looking at the reduced echelon form of the matrix, we see that it is of rank 2 and that a particular solution of the given equation is
\[
\begin{bmatrix}
  u \\
  v
\end{bmatrix} = \begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix}.
\]

7. Problem 2.2.20. *If \( A \) has rank \( r \), then it has an \( r \) by \( r \) submatrix \( S \) that is invertible.* Find that submatrix \( S \) from the pivot rows and pivot columns of each \( A \):

\[
A = \begin{bmatrix}
  1 & 2 & 3 \\
  1 & 2 & 4
\end{bmatrix} \quad A = \begin{bmatrix}
  1 & 2 & 3 \\
  2 & 4 & 6
\end{bmatrix} \quad A = \begin{bmatrix}
  0 & 1 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 1
\end{bmatrix}.
\]

**Answer:** For the first matrix, if we subtract row 1 from row 2 we get the reduced matrix
\[
\begin{bmatrix}
  1 & 2 & 3 \\
  0 & 0 & 1
\end{bmatrix},
\]
so we see that the pivot columns are the first and third columns, and the pivot rows are the first and second rows. Hence, the invertible 2 by 2 submatrix of \( A \) consists of the first and third columns of the first and second rows, namely
\[
\begin{bmatrix}
  1 & 3 \\
  1 & 4
\end{bmatrix}.
\]

For the second choice of \( A \), subtracting row 1 from row 2 yields the reduced matrix
\[
\begin{bmatrix}
  1 & 2 & 3 \\
  0 & 0 & 0
\end{bmatrix}.
\]

Hence, the first column is the only pivot column and the first row is the only pivot row. Therefore, the rank of \( A \) is 1 and the invertible 1 by 1 submatrix consists of the first column of the first row, namely
\[
\begin{bmatrix} 1 \end{bmatrix}.
\]

For the third choice of \( A \), we don’t have to do any elimination to see that the pivot columns of \( A \) are the second and third columns and the pivot rows of \( A \) are the first and third rows.
Hence, $A$ has rank 2 and the invertible 2 by 2 submatrix consists of the second and third columns of the first and third rows, namely

$$
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
$$

8. Problem 2.2.30. Execute the six steps following equation (6) to find the column space and nullspace of $A$ and the solution to $Ax = b$:

$$
A = \begin{bmatrix}
1 & 1 & 2 & 2 \\
2 & 5 & 7 & 6 \\
2 & 3 & 5 & 2
\end{bmatrix}, \quad b = \begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix} = \begin{bmatrix}
4 \\
3 \\
5
\end{bmatrix}.
$$

**Answer:** Form the augmented matrix $[A \ b]$:

$$
\begin{bmatrix}
2 & 4 & 6 & 4 & 4 \\
2 & 5 & 7 & 6 & 3 \\
2 & 3 & 5 & 2 & 5
\end{bmatrix}.
$$

Then subtracting row 1 from rows 2 and 3 and multiplying row 1 by $\frac{1}{2}$ yields

$$
\begin{bmatrix}
1 & 2 & 3 & 2 & 2 \\
0 & 1 & 1 & 2 & -1 \\
0 & -1 & -1 & -2 & 1
\end{bmatrix}.
$$

Next, subtracting twice row 2 from row 1 and adding row 2 to row 3 gives

$$
\begin{bmatrix}
1 & 0 & 1 & -2 & 4 \\
0 & 1 & 1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

This is now in reduced echelon form, so we can answer the question. Notice that the pivot columns are the first and second columns; hence, the column space of $A$ is the span of the first two columns of $A$, namely $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$. Geometrically, this is just the plane containing these two vectors.

Returning the reduced echelon form of the augmented matrix, notice that we must have

$$
x_1 = 4 - x_3 + 2x_4 \\
x_2 = -1 - x_3 - 2x_4,
$$

so the special solutions are of the form

$$
x_3 \begin{bmatrix}
-1 \\
-1 \\
0
\end{bmatrix} + x_4 \begin{bmatrix}
2 \\
-2 \\
1
\end{bmatrix}.
$$
for some real numbers $x_3$ and $x_4$. Hence, the nullspace of $A$ consists precisely of such linear combinations.

Finally, all solutions to the equation $Ax = b$ are of the form

$$\begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix},$$

where the first term is a particular solution and the latter two terms comprise the special (or homogeneous) solutions.

9. Problem 2.2.62. Construct a matrix whose column space contains $(1, 1, 5)$ and $(0, 3, 1)$ and whose nullspace contains $(1, 1, 2)$.

**Answer:** The simplest way of constructing a matrix whose column space contains a given vector is to make that vector a column of the matrix. Hence, let

$$A = \begin{bmatrix} 1 & 0 & a_1 \\ 1 & 3 & a_2 \\ 5 & 1 & a_3 \end{bmatrix}.$$  

Then the column space of $A$ automatically contains the two desired vectors, and we just need to find $a_1, a_2, a_3$ such that $(1, 1, 2)$ is in the nullspace. But this just means that we need to choose $a_1, a_2, a_3$ such that

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a_1 \\ 1 & 3 & a_2 \\ 5 & 1 & a_3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 + a_1 \\ 4 + 2a_2 \\ 6 + 2a_3 \end{bmatrix}.$$  

Hence, we can pick $a_1 = -1/2$, $a_2 = -2$, and $a_3 = -3$, so the matrix

$$A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$$

has all of the desired properties.

10. Suppose $x_p$ is a vector in $\mathbb{R}^n$ such that

$$Ax_p = b,$$

where $A$ is a given $m \times n$ matrix and $b$ is a given vector in $\mathbb{R}^m$. Prove that, if $x$ is any solution to the equation $Ax = b$, then

$$x = x_p + x_h,$$

where $x_h$ is some element of the nullspace of $A$. 

8
Proof. Suppose $x \in \mathbb{R}^n$ such that $Ax = b$. The goal is to find $x_h$ such that $x = x_p + x_h$.

In search of that $x_h$, notice that

$$A(x - x_p) = Ax - Ax_p = b - b = 0.$$ 

Hence, $x - x_p$ is in the nullspace of $A$. Letting $x_h = x - x_p$, we see that

$$x_p + x_h = x_p + (x - x_p) = x,$$

as desired. \qed