Math 215 HW #11 Solutions

1. Problem 5.5.6. Find the lengths and the inner product of

\[ \vec{x} = \begin{bmatrix} 2 - 4i \\ 4i \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} 2 + 4i \\ 4i \end{bmatrix}. \]

**Answer:** First,

\[ \|\vec{x}\|^2 = \vec{x}^H \vec{x} = [2 + 4i \quad - 4i] \begin{bmatrix} 2 - 4i \\ 4i \end{bmatrix} = (4 + 16) + 16 = 36, \]

so \(\|\vec{x}\| = 6\). Likewise,

\[ \|\vec{y}\|^2 = \vec{y}^H \vec{y} = [2 - 4i \quad - 4i] \begin{bmatrix} 2 + 4i \\ 4i \end{bmatrix} = (4 + 16) + 16, \]

so \(\|\vec{y}\| = 6\).

Finally,

\[ \langle \vec{x}, \vec{y} \rangle = \vec{x}^H \vec{y} = [2 + 4i \quad - 4i] \begin{bmatrix} 2 + 4i \\ 4i \end{bmatrix} = (2 + 4i)^2 - (4i)^2 = (4 - 16 + 16i) + 16 = 4 + 16i. \]

2. Problem 5.5.16. Write one significant fact about the eigenvalues of each of the following:

(a) A real symmetric matrix.

**Answer:** As we saw in class, the eigenvalues of a real symmetric matrix are all real numbers.

(b) A stable matrix: all solutions to \(du/dt = Au\) approach zero.

**Answer:** By the definition of stability, this means that the reals parts of the eigenvalues of \(A\) are non-positive.

(c) An orthogonal matrix.

**Answer:** If \(A\vec{x} = \lambda \vec{x}\), then

\[ \langle A\vec{x}, A\vec{x} \rangle = \langle \lambda \vec{x}, \lambda \vec{x} \rangle = \lambda^2 \langle \vec{x}, \vec{x} \rangle = \lambda^2 \|\vec{x}\|^2. \]

On the other hand,

\[ \langle A\vec{x}, A\vec{x} \rangle = (A\vec{x})^T A\vec{x} = \vec{x}^T A^T A\vec{x} = \vec{x}^T \vec{x} = \langle \vec{x}, \vec{x} \rangle = \|\vec{x}\|^2. \]

Therefore,

\[ \lambda^2 \|\vec{x}\|^2 - \|\vec{x}\|^2, \]

meaning that \(\lambda^2 = 1\), so \(|\lambda| = 1\).

(d) A Markov matrix.

**Answer:** We saw in class that \(\lambda_1 = 1\) is an eigenvalue of every Markov matrix, and that all eigenvalues \(\lambda_i\) of a Markov matrix satisfy \(|\lambda_i| \leq 1\).
(e) A defective matrix (nondiagonalizable).

**Answer:** If \( A \) is \( n \times n \) and is not diagonalizable, then \( A \) must have fewer than \( n \) eigenvalues (if \( A \) had \( n \) distinct eigenvalues and since eigenvectors corresponding to different eigenvalues are linear independent, then \( A \) would have \( n \) linearly independent eigenvectors, which would imply that \( A \) is diagonalizable).

(f) A singular matrix.

**Answer:** If \( A \) is singular, then \( A \) has a non-trivial nullspace, which means that 0 must be an eigenvalue of \( A \).

3. Problem 5.5.22. Every matrix \( Z \) can be split into a Hermitian and a skew-Hermitian part, \( Z = A + K \), just as a complex number \( z \) is split into \( a + ib \). The real part of \( z \) is half of \( z + \overline{z} \), and the “real part” (i.e. Hermitian part) of \( Z \) is half of \( Z + Z^H \). Find a similar formula for the “imaginary part” (i.e. skew-Hermitian part) \( K \), and split these matrices into \( A + K \):

\[
Z = \begin{bmatrix}
3 + 4i & 4 + 2i \\
0 & 5
\end{bmatrix}
\quad \text{and} \quad
Z = \begin{bmatrix}
i & i \\
-i & i
\end{bmatrix}.
\]

**Answer:** Notice that

\[
(Z + Z^H)^H = Z^H + (Z^H)^H = Z^H + Z,
\]

so indeed \( \frac{1}{2}(Z + Z^H) \) is Hermitian. Likewise,

\[
(Z - Z^H)^H = Z^H - (Z^H)^H = Z^H - Z = -(Z - Z^H),
\]

is skew-Hermitian, so \( K = \frac{1}{2}(Z - Z^H) \) is the skew-Hermitian part of \( Z \).

Hence, when

\[
Z = \begin{bmatrix}
3 + 4i & 4 + 2i \\
0 & 5
\end{bmatrix},
\]

we have

\[
A = \frac{1}{2}(Z + Z^H) = \frac{1}{2} \left( \begin{bmatrix}
3 + 4i & 4 + 2i \\
0 & 5
\end{bmatrix} + \begin{bmatrix}
3 - 4i & 0 \\
4 - 2i & 5
\end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix}
6 & 4 + 2i \\
4 - 2i & 10
\end{bmatrix} = \begin{bmatrix}
3 & 4 + 2i \\
4 - 2i & 5
\end{bmatrix}
\]

and

\[
K = \frac{1}{2}(Z - Z^H) = \frac{1}{2} \left( \begin{bmatrix}
3 + 4i & 4 + 2i \\
0 & 5
\end{bmatrix} - \begin{bmatrix}
3 - 4i & 0 \\
4 - 2i & 5
\end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix}
8i & 4 + 2i \\
-4 + 2i & 0
\end{bmatrix} = \begin{bmatrix}
4i & 4 + 2i \\
-4 + 2i & 0
\end{bmatrix}.
\]

On the other hand, when

\[
Z = \begin{bmatrix}
i & i \\
-i & i
\end{bmatrix}.
\]

we have

\[
A = \frac{1}{2}(Z + Z^H) = \frac{1}{2} \left( \begin{bmatrix}
i & i \\
-i & i
\end{bmatrix} + \begin{bmatrix}
-i & i \\
-i & -i
\end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix}
0 & 2i \\
-2i & 0
\end{bmatrix} = \begin{bmatrix}
0 & i \\
-i & 0
\end{bmatrix}
\]

and

\[
K = \frac{1}{2}(Z - Z^H) = \frac{1}{2} \left( \begin{bmatrix}
i & i \\
-i & i
\end{bmatrix} - \begin{bmatrix}
-i & i \\
-i & -i
\end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix}
2i & 0 \\
0 & 2i
\end{bmatrix} = \begin{bmatrix}
i & 0 \\
0 & i
\end{bmatrix}.
\]
4. Problem 5.5.28. If \( \vec{A}z = \vec{0} \), then \( \vec{A}H\vec{A}z = \vec{0} \). If \( \vec{A}H\vec{A}z = \vec{0} \), multiply by \( \vec{z}H \) to prove that \( \vec{A}z = \vec{0} \). The nullspaces of \( A \) and \( AH \) are \( \underline{_____} \). \( A^H A \) is an invertible Hermitian matrix when the nullspace of \( A \) contains only \( \vec{z} = \underline{\vec{0}} \).

Answer: Suppose \( A^H A\vec{z} = \vec{0} \). Then, multiplying both sides by \( \vec{z}H \) yields
\[
0 = \vec{z}H A^H A\vec{z} = (A\vec{z})H(A\vec{z}) = \langle A\vec{z}, A\vec{z} \rangle = \| A\vec{z} \|_2^2,
\]
meaning that \( A\vec{z} = \vec{0} \).

Therefore, we see that if \( A\vec{z} = \vec{0} \), then \( A^H A\vec{z} = \vec{0} \) and if \( A^H A\vec{z} = \vec{0} \), then \( A\vec{z} = \vec{0} \), so the nullspaces of \( A \) and \( A^H \) are equal.

5. Problem 5.5.48. Prove that the inverse of a Hermitian matrix is again a Hermitian matrix.

Proof. If \( A \) is Hermitian, then
\[
A = U\Lambda U^H,
\]
where \( U \) is unitary and \( \Lambda \) is a real diagonal matrix. Therefore,
\[
A^{-1} = (U\Lambda U^H)^{-1} = (U^H)^{-1}\Lambda^{-1}U^{-1} = U\Lambda^{-1}U^H
\]
since \( U^{-1} = U^H \). Note that \( \Lambda^{-1} \) is just the diagonal matrix with entries \( 1/\lambda_i \) (where the \( \lambda_i \) are the entries in \( \Lambda \)). Hence,
\[
(A^{-1})^H = (U\Lambda^{-1}U^H)^H = U(\Lambda^{-1})^H U^H = U\Lambda^{-1}U^H = A^{-1}
\]
since \( \Lambda^{-1} \) is a real matrix, so we see that \( A^{-1} \) is Hermitian. \( \Box \)

6. Problem 5.6.8. What matrix \( M \) changes the basis \( \vec{V}_1 = (1, 1), \vec{V}_2 = (1, 4) \) to the basis \( \vec{v}_1 = (2, 5), \vec{v}_2 = (1, 4) \)? The columns of \( M \) come from expressing \( \vec{V}_1 \) and \( \vec{V}_2 \) as combinations \( \sum m_{ij}\vec{v}_i \) of the \( \vec{v} \)'s.

Answer: Since
\[
\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \vec{v}_1 - \vec{v}_2
\]
and
\[
\vec{V}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \vec{v}_2,
\]
we see that
\[
M = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.
\]

7. Problem 5.6.12. The identity transformation takes every vector to itself: \( T\vec{x} = \vec{x} \). Find the corresponding matrix, if the first basis is \( \vec{v}_1 = (1, 2), \vec{v}_2 = (3, 4) \) and the second basis is \( \vec{w}_1 = (1, 0), \vec{w}_2 = (0, 1) \). (It is not the identity matrix!)

Answer: Despite the slightly confusing way this question is worded, it is just asking for the matrix \( M \) which converts the \( \vec{v} \) basis into the \( \vec{w} \) basis. Clearly,
\[
\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{w}_1 + 2\vec{w}_2
\]
and
\[ \vec{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3 \vec{w}_1 + 4 \vec{w}_2, \]
so the desired matrix is
\[ M = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}. \]

8. Problem 5.6.38. These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors (find them). But the block sizes don’t match and \( J \) is not similar to \( K \).

\[ J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]

For any matrix \( M \), compare \( JM \) and \( MK \). If they are equal, show that \( M \) is not invertible. Then \( M^{-1}JM = K \) is impossible.

**Answer:** First, we find the eigenvectors of \( J \) and \( K \). Since all eigenvalues of both are 0, we’re just looking for vectors in the nullspace of \( J \) and \( K \). First, for \( J \), we note that \( J \vec{v} = \vec{0} \) implies that \( \vec{v} \) is a linear combination of
\[ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \]

Hence, these are the eigenvectors of \( J \).

Likewise, \( K \) is already in reduced echelon form and \( K \vec{v} = \vec{0} \) implies that \( \vec{v} \) is a linear combination of
\[ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \]

Hence, these are the eigenvectors of \( K \).

Now, suppose
\[ M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \]
such that \( JM = MK \). Then
\[ JM = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} = \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]
and
\[
MK = \begin{bmatrix}
m_{11} & m_{12} & m_{13} & m_{14} \\
m_{21} & m_{22} & m_{23} & m_{24} \\
m_{31} & m_{32} & m_{33} & m_{34} \\
m_{41} & m_{42} & m_{43} & m_{44}
\end{bmatrix} \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & m_{11} & m_{12} & 0 \\
0 & m_{21} & m_{22} & 0 \\
0 & m_{31} & m_{32} & 0 \\
0 & m_{41} & m_{42} & 0
\end{bmatrix}.
\]

Therefore \(JM = MK\) means that
\[
\begin{bmatrix}
m_{21} & m_{22} & m_{23} & m_{24} \\
0 & 0 & 0 & 0 \\
m_{41} & m_{42} & m_{43} & m_{44} \\
0 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & m_{11} & m_{12} & 0 \\
0 & m_{21} & m_{22} & 0 \\
0 & m_{31} & m_{32} & 0 \\
0 & m_{41} & m_{42} & 0
\end{bmatrix}
\]
and so we have that
\[m_{21} = m_{24} = m_{41} = m_{44}\]

Plugging these back into \(M\), we see that
\[
M = \begin{bmatrix}
m_{11} & m_{12} & m_{13} & m_{14} \\
0 & 0 & m_{23} & 0 \\
m_{31} & m_{32} & m_{33} & m_{34} \\
0 & 0 & m_{43} & 0
\end{bmatrix}.
\]

Clearly, the second and fourth rows are multiples of each other, so \(M\) cannot possibly have rank 4. However, \(M\) not having rank 4 means that \(M\) cannot be invertible. Therefore, \(M^{-1}JM = K\) is impossible, so it cannot be the case that \(J\) and \(K\) are similar.

9. Problem 5.6.40. Which pairs are similar? Choose \(a, b, c, d\) to prove that the other pairs aren’t:
\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \begin{bmatrix}
b & a \\
d & c
\end{bmatrix} \begin{bmatrix}
c & d \\
a & b
\end{bmatrix} \begin{bmatrix}
d & c \\
b & a
\end{bmatrix}.
\]

**Answer:** The second and third are clearly similar, since
\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}^{-1} = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
b & a \\
d & c
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} = \begin{bmatrix}
c & d \\
a & b
\end{bmatrix}.
\]
Likewise, the first and fourth are similar, since
\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
d & c \\
b & a
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
c & d \\
a & b
\end{bmatrix} = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}.
\]
There are no other similarities, as we can see by choosing
\[a = 1, \quad b = c = d = 0.\]
Then the matrices are, in order
\[
\begin{bmatrix}
1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]
Each of these is already a diagonal matrix, and clearly the first and fourth have 1 as an eigenvalue, whereas the second and third have only 0 as an eigenvalue. Since similar matrices have the same eigenvalues, we see that neither the first nor the fourth can be similar to either the second or the third.

10. **(Bonus Problem)** Problem 5.6.14. Show that every number is an eigenvalue for \( T f(x) = df/dx \), but the transformation \( T f(x) = \int_0^x f(t)dt \) has no eigenvalues (here \(-\infty < x < \infty\)).

**Proof.** For the first \( T \), note that, if \( f(x) = e^{ax} \) for any real number \( a \), then
\[ T f(x) = \frac{df}{dx} = ae^{ax} = af(x). \]
Hence, any real number \( a \) is an eigenvalue of \( T \).

Turning to the second \( T \), suppose we had that \( T f(x) = af(x) \) for some number \( a \) and some function \( f \). Then, by the definition of \( T \),
\[ \int_0^x f(t)dt = af(x). \]
Now, use the fundamental theorem of calculus to differentiate both sides:
\[ f(x) = af'(x). \]
Solving for \( f \), we see that
\[ \int f'(x)dx \frac{f(x)}{f(x)} = \int \frac{1}{a}dx, \]
so
\[ \ln |f(x)| = \frac{x}{a} + C. \]
Therefore, exponentiating both sides,
\[ |f(x)| = e^{x/a+C} = e^C e^{x/a}. \]
We can get rid of the absolute value signs by substituting \( A \) for \( e^C \) (allowing \( A \) to possibly be negative):
\[ f(x) = Ae^{x/a}. \]
Therefore, we know that
\[ T f(x) = \int_0^x f(t)dt = \int_0^x Ae^{t/a}dt = aAe^{x/a} = aAf(x) = a(f(x) - A). \]
On the other hand, our initial assumption was that \( T f(x) = af(x) \), so it must be the case that
\[ af(x) = a(f(x) - A) = af(x) - aA. \]
Hence, either \( a = 0 \) or \( A = 0 \). However, either implies that \( f(x) = 0 \), so \( T \) has no eigenvalues. \( \square \)