1. For each of the following statements, say whether it is true or false. If the statement is true, prove it. If false, give a counterexample.
   
   (a) If eigenvectors \( \vec{x} \) and \( \vec{y} \) correspond to distinct eigenvalues, then \( \vec{x}^T \vec{y} = 0 \).
   
   (b) Let \( A \) be an \( m \times n \) matrix and let \( \vec{b} \) be a vector in \( \mathbb{R}^m \). If \( m < n \), then \( A\vec{x} = \vec{b} \) has infinitely many solutions.
   
   (c) If \( A \) is an \( m \times n \) real matrix, then the nullspace of \( A^T \) is the orthogonal complement of the column space of \( A \).
   
   (d) If \( S \) and \( T \) are subspaces of \( \mathbb{R}^2 \), then their union (i.e., the set of vectors which are in either \( S \) or \( T \)) is also a subspace of \( \mathbb{R}^2 \).
   
   (e) Let \( S \) be a plane through the origin in \( \mathbb{R}^3 \) and let \( P \) be the matrix which projects onto the plane \( S \). Then for any \( \vec{v} \in \mathbb{R}^3 \),
   
   \[
   \| \vec{v} \|^2 = \| P\vec{v} \|^2 + \| \vec{v} - P\vec{v} \|^2.
   \]

2. Is the set of all orthogonal \( n \times n \) real matrices a vector space?

3. Suppose the first row of \( A \) is 2, 3 and its eigenvalues are \( i, -i \). Find \( A \).

4. If \( \vec{x}_1, \vec{x}_2 \) are the columns of \( S \), what are the eigenvalues and eigenvectors of

   \[
   A = S \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} S^{-1} \quad \text{and} \quad B = S \begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix} S^{-1}?
   \]

5. If \( A \) is an \( n \times (n - 1) \) matrix with rank \( n - 2 \) and \( \vec{b} \in \mathbb{R}^n \) such that \( A\vec{v} = \vec{b} \) for some \( \vec{v} \in \mathbb{R}^{n-1} \), what is the dimension of the space of all solutions to the equation \( A\vec{x} = \vec{b} \)?

6. (a) Prove that the eigenvalues of \( A \) are the eigenvalues of \( A^T \).
   
   (b) If \( A \) and \( B \) are \( n \times n \) real symmetric matrices, then \( AB \) and \( BA \) have the same eigenvalues. [HINT: Use part (a)]

7. Is there a real \( 2 \times 2 \) matrix \( A \neq I \) such that \( A^3 = I \)?

8. Suppose \( A \) is diagonalizable and that

   \[
   p(\lambda) = \det(A - \lambda I) = c_n \lambda^n + c_{n-1} \lambda^{n-1} + \ldots + c_1 \lambda + c_0.
   \]

   Show that

   \[
   p(A) = c_n A^n + c_{n-1} A^{n-1} + \ldots + c_1 A + c_0 I
   \]

   is the zero matrix.

   NOTE: The fact that this is true for any matrix (regardless of whether it’s diagonalizable) is called the Cayley-Hamilton Theorem.

9. Suppose \( A = \vec{u}\vec{v}^T \) where \( \vec{u}, \vec{v} \in \mathbb{R}^n \) and \( \vec{v} \neq \vec{0} \).

   (a) What is the rank of \( A \)?
   
   (b) Show that \( \vec{u} \) is an eigenvector of \( A \). What is the corresponding eigenvalue?
   
   (c) What are the other eigenvalues of \( A \)?
   
   (d) Compute the trace of \( A \) in two different ways.
10. Let $V$ be an $n$-dimensional vector space and suppose $T : V \to V$ is a linear transformation such that the range of $T$ is equal to the set of vectors that $T$ sends to $\vec{0}$. In other words,

$$\{T(\vec{v}) : \vec{v} \in V\} = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}.$$

(a) Show that $n$ must be even.
(b) Give an example of such a $T$.

11. Find the $LU$ decomposition of the matrix

$$
\begin{bmatrix}
3 & -1 & 2 \\
-3 & -2 & 10 \\
9 & -5 & 6 \\
\end{bmatrix}
$$

12. Let $A$ be the matrix

$$A = \begin{bmatrix}
-2 & 2 \\
1 & -1 \\
\end{bmatrix}
$$

and suppose $\vec{u}(t)$ solves the differential equation

$$\frac{d\vec{u}}{dt} = A\vec{u}(t)
$$

subject to the initial condition $\vec{u}(0) = \begin{bmatrix}1 \\ 4 \end{bmatrix}$. What happens to $\vec{u}(t)$ as $t \to \infty$?