1. For each of the following statements, say whether it is true or false. If the statement is true, prove it. If false, give a counterexample.
   (a) If \( Q \) is an orthogonal matrix, then \( \det Q = 1 \).
   (b) Every invertible matrix can be diagonalized.
   (c) Every diagonalizable matrix is invertible.
   (d) If the matrix \( A \) is not invertible, then 0 is an eigenvalue of \( A \).
   (e) If \( \vec{v} \) and \( \vec{w} \) are orthogonal and \( P \) is a projection matrix, then \( P\vec{v} \) and \( P\vec{w} \) are also orthogonal.
   (f) Suppose \( A \) is an \( n \times n \) matrix and that there exists some \( k \) such that \( A^k = 0 \) (such matrices are called nilpotent matrices). Then \( A \) is not invertible.

2. Let \( Q \) be an \( n \times n \) orthogonal matrix. Show that if \( \{\vec{v}_1, \ldots, \vec{v}_n\} \) is an orthonormal basis for \( \mathbb{R}^n \), then so is \( \{Q\vec{v}_1, \ldots, Q\vec{v}_n\} \).

3. Let \( A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \).
   (a) Let \( R \) be the region in the plane enclosed by the unit circle. If \( T \) is the linear transformation of the plane whose matrix is \( A \), what is the area of \( T(R) \)?
   (b) Find the matrix for the transformation \( T^{-1} \) without doing elimination.

4. Let \( \ell \) be the line in \( \mathbb{R}^3 \) through the vector \( \vec{a} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \).
   (a) Find a basis for the orthogonal complement of \( \ell \).
   (b) If \( \vec{v} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \), write \( \vec{v} \) as a sum
       \[ \vec{v} = \vec{v}_1 + \vec{v}_2, \]
       where \( \vec{v}_1 \in \ell \) and \( \vec{v}_2 \in \ell^\perp \).

5. Find the line \( C + Dt \) that best fits the data \((-1,1), (0,1), (1,2)\).

6. Let \( \ell \) be the line through a vector \( \vec{a} \in \mathbb{R}^n \) and let \( P \) be the matrix which projects everything in \( \mathbb{R}^n \) to \( \ell \).
   (a) Show that the trace of \( P \) equals 1.
   (b) What can you say about the eigenvalues of \( P \)?

7. Suppose \( A \) is a \( 2 \times 2 \) matrix with eigenvalues \( \lambda_1 \) and \( \lambda_2 \) corresponding to non-zero eigenvectors \( \vec{v}_1 \) and \( \vec{v}_2 \), respectively. If \( \lambda_1 \neq \lambda_2 \), show that \( \vec{v}_1 \) and \( \vec{v}_2 \) are linearly independent.