Math 215 Exam #1 Practice Problems

1. For each of the following statements, say whether it is true or false. If the statement is true, prove it. If false, give a counterexample.

(a) If \(A\) is a \(2 \times 2\) matrix such that \(A(x) = 0\) for all \(x \in \mathbb{R}^2\), then \(A\) is the zero matrix.
(b) A system of 3 equations in 4 unknowns can never have a unique solution.
(c) If \(V\) is a vector space and \(S\) is a finite set of vectors in \(V\), then some subset of \(S\) forms a basis for \(V\).
(d) Suppose \(A\) is an \(m \times n\) matrix such that \(Ax = b\) can be solved for any choice of \(b \in \mathbb{R}^m\). Then the columns of \(A\) form a basis for \(\mathbb{R}^m\).
(e) Given 3 equations in 4 unknowns, each describes a hyperplane in \(\mathbb{R}^4\). If the system of those 3 equations is consistent, then the intersection of the hyperplanes contains a line.
(f) If \(A\) is a symmetric matrix (i.e. \(A = A^T\)), then \(A\) is invertible.
(g) If \(m < n\) and \(A\) is an \(m \times n\) matrix such that \(Ax = b\) has a solution for all \(b \in \mathbb{R}^m\), then there exists \(z \in \mathbb{R}^m\) such that \(Ax = z\) has infinitely many solutions.
(h) The set of polynomials of degree \(\leq 5\) forms a vector space.

2. For each of the following, determine whether the given subset is a subspace of the given vector space. Explain your answer.

(a) **Vector Space:** \(\mathbb{R}^4\).
   **Subset:** The vectors of the form
   \[
   \begin{bmatrix}
   a \\
   b \\
   0 \\
   d
   \end{bmatrix}
   .
   
   (b) **Vector Space:** \(\mathbb{R}^2\).
   **Subset:** The solutions to the equation \(2x - 5y = 11\).

   (c) **Vector Space:** \(\mathbb{R}^n\).
   **Subset:** All \(x \in \mathbb{R}^n\) such that \(Ax = 2x\) where \(A\) is a given \(n \times n\) matrix.

   (d) **Vector Space:** \(\mathbb{R}^3\).
   **Subset:** The intersection of \(P_1\) and \(P_2\), where \(P_1\) and \(P_2\) are planes through the origin.

   (e) **Vector Space:** All polynomials.
   **Subset:** The quadratic (i.e. degree 2) polynomials.

   (f) **Vector Space:** All real-valued functions.
   **Subset:** Functions of the form \(f(t) = a \cos t + b \sin t + c\) for \(a, b, c \in \mathbb{R}\).

3. Consider the matrix
   \[
   A = \begin{bmatrix}
   1 & a \\
   a & 1
   \end{bmatrix}.
   
   (a) Under what conditions on \(a\) is \(A\) invertible?
   (b) Choose a non-zero value of \(a\) that makes \(A\) invertible and determine \(A^{-1}\).
   (c) For each value of \(a\) that makes \(A\) non-invertible, determine the dimension of the nullspace of \(A\).
4. Consider the system of equations

\[
\begin{align*}
   x_1 + 2x_2 + x_3 - 3x_4 &= b_1 \\
   x_1 + 2x_2 + 2x_3 - 5x_4 &= b_2 \\
   2x_1 + 4x_2 + 3x_3 - 8x_4 &= b_3
\end{align*}
\]

(a) Find all solutions when the above system is homogeneous (i.e. \( b_1 = b_2 = b_3 = 0 \)). Find a basis for the space of solutions to the homogeneous system.

(b) Let \( S \) be the set of vectors \( \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \) such that the system can be solved. What is the dimension of \( S \)?

(c) It’s easy to check that the vector \( \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \) is a solution to the system that arises when \( b_1 = 3, b_2 = 5, \) and \( b_3 = 8 \). Find all the solutions to this system.