Math 115 HW #4 Solutions

From §12.5

8. Does the series
\[ \sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3} + 2} \]
converge or diverge?

**Answer:** This is an alternating series, so we need to check that the terms satisfy the hypotheses of the Alternating Series Test. To see that the terms are decreasing in absolute value (i.e. that \( b_{n+1} < b_n \)), define
\[ f(x) = \frac{x}{\sqrt{x^3} + 2}. \]

Then
\[ f'(x) = \frac{\sqrt{x^3} + 2 - \frac{x}{2\sqrt{x^3} + 2} \cdot 3x^2}{x^2} = \frac{2(x^3 + 2) - 3x^3}{2x^2 \sqrt{x^3} + 2} = \frac{4 - x^3}{2x^2 \sqrt{x^3} + 2}. \]

So long as \( x > 0 \), the term on the right is negative, so we see that \( f \) is a decreasing function. Therefore, the terms of the sequence are decreasing in absolute value.

To see that the terms are going to zero, we need to show that
\[ \lim_{n \to \infty} \frac{n}{\sqrt{n^3} + 2} = 0. \]

In the lefthand side, multiply both numerator and denominator by \( \frac{1}{n} \). This yields
\[ \lim_{n \to \infty} \frac{1}{n} \sqrt{\frac{1}{n^3} + \frac{2}{n}} = \lim_{n \to \infty} \frac{1}{\sqrt{n^3 + 2/n}}. \]

Since the numerator is constant and the denominator goes to infinity as \( n \to \infty \), this limit is equal to zero.

Therefore, we can apply the Alternating Series Test, which says that the series converges.

12. Does the series
\[ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{1/n}}{n} \]
converge or diverge?

**Answer:** Again, we want to use the Alternating Series Test, so we need to confirm that the terms are decreasing in absolute value and going to zero.

To see that the terms are decreasing, we want to show that
\[ \frac{e^{1/n+1}}{n+1} < \frac{e^{1/n}}{n}. \]
in other words, we want to show that
\[ \frac{n+\sqrt{e}}{n+1} < \frac{\sqrt{e}}{n}. \]
But certainly \( n+\sqrt{e} < \sqrt{e} \) and \( n+1 > n \), so the above inequality is true, which implies that the first inequality is true and that the terms are decreasing in absolute value.

To see that the terms are going to zero, we just examine the limit
\[ \lim_{n \to \infty} \frac{\frac{e\pi}{n}}{n} = \lim_{n \to \infty} \frac{\sqrt{e}}{n} = 0, \]
since \( \lim \lfloor n \rfloor e = 1 \).

Therefore, by the Alternating Series Test, the series converges.

16. Does the series
\[ \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n!} \]
converge or diverge?

**Answer:** This series does converge, but, we’ll see that the slightly less complicated series
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \]
converges.

To use the Alternating Series Test, we need to show that the terms are decreasing in absolute value, so we need to show that
\[ \frac{1}{(n+1)!} < \frac{1}{n!}. \]
Multiplying both sides by \((n+1)!\) shows that this inequality is equivalent to the inequality
\[ 1 < \frac{(n+1)!}{n!} = \frac{n+1}{1} = n+1, \]
which is certainly true. Therefore, the original inequality is true and the terms are decreasing in absolute value.

Also,
\[ \lim_{n \to \infty} \frac{1}{n!} = 0 \]
since \( 0 < \frac{1}{n!} < \frac{1}{n} \) and \( \lim_{n \to \infty} \frac{1}{n} \to 0 \).

Therefore, the Alternating Series Test tells us that the series converges.

24. Show that the series
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n}}{n5^{n}} \]
is convergent. How many terms of the series do we need to add in order to approximate the sum of the series to within 0.0001?

**Answer:** To see that the terms are decreasing in absolute value we need to show that

$$\frac{1}{(n+1)5^{n+1}} < \frac{1}{n5^n}.$$  

Multiplying by $(n+1)5^{n+1}$, we see that this inequality is equivalent to

$$1 < \frac{(n+1)5^{n+1}}{n5^n} = \frac{5n+1}{n},$$

which is clearly true since $\frac{n+1}{n} > 1$. Hence, the terms are decreasing in absolute value. To see that the limit of the terms is zero, just take

$$\lim_{n \to \infty} \frac{1}{n5^n} = 0,$$

since the numerator is constant and the denominator $\to \infty$.

Therefore, by the Alternating Series Test, the series converges.

If we want the error to be less than 0.0001, then we need to find $n$ such that $R_n = s - s_n$ is bounded by

$$|R_n| \leq 0.0001.$$  

However, we know that, for alternating series, the remainder is bounded as

$$|R_n| \leq b_{n+1} = \frac{1}{(n+1)5^{n+1}}.$$  

Therefore, if we can find $n$ such that $\frac{1}{(n+1)5^{n+1}} < 0.0001$, we will have that

$$|R_n| \leq \frac{1}{(n+1)5^{n+1}} < 0.0001,$$

as desired.

But clearly, if $n = 4$, then $(n+1)5^{n+1} = 5 \cdot 5^5 = 15,625 > 10,000$, meaning that

$$\frac{1}{(n+1)5^{n+1}} < \frac{1}{10,000} = 0.0001,$$

so we need only add the first 4 terms of the series to approximate the sum within 0.0001.

**32.** For what values of $p$ is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$$

convergent?

**Answer:** When $p < 0$, the terms are blowing up, so the series diverges. When $p = 0$, we have $\sum(-1)^{n-1}$, which also diverges. I claim that when $p > 0$ the series converges.
To see this, I’ll use the Alternating Series Test, which requires that the terms decrease in absolute value and that they go to zero. To see that they decrease in absolute value, I need to show that
\[
\frac{1}{(n+1)^p} < \frac{1}{n^p}.
\]
Multiplying both sides by \((n + 1)^p\), this is equivalent to the inequality
\[
1 < \frac{(n + 1)^p}{n^p} = \left(\frac{n + 1}{n}\right)^p,
\]
which is certainly true since \(\frac{n+1}{n} > 1\).
To see that the terms go to zero, consider the limit
\[
\lim_{n \to \infty} \frac{1}{n^p}.
\]
This limit is certainly zero since the numerator is constant and the denominator is going to \(\infty\) (because \(p > 0\)).
Therefore, the Alternating Series Test tells us that \(\sum \frac{(-1)^{n-1}}{n^p}\) converges for \(p > 0\).

From §12.6

12. Is the series
\[
\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}
\]
absolutely convergent, conditionally convergent, or divergent?
Answer: Using the fact that \(|\sin x| \leq 1\) for any \(x\), we know that
\[
\left|\frac{\sin 4n}{4^n}\right| = \frac{|\sin 4n|}{4^n} \leq \frac{1}{4^n}.
\]
Since the series
\[
\sum_{n=1}^{\infty} \frac{1}{4^n}
\]
converges (since it’s a geometric series), we know, by the Comparison Test, that
\[
\sum_{n=1}^{\infty} \left|\frac{\sin 4n}{4^n}\right|
\]
converges. Thus, the series \(\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}\) converges absolutely.

18. Is the series
\[
\sum_{n=1}^{\infty} \frac{n!}{n^n}
\]
absolutely convergent, conditionally convergent, or divergent?
Answer: Using the Ratio Test,
\[
\lim_{n \to \infty} \left| \frac{(n+1)!}{n!} \frac{n!}{n^{n+1}} \right| = \lim_{n \to \infty} \frac{n(n+1)!}{(n+1)n^{n+1}} = \lim_{n \to \infty} \frac{n^n}{(n+1)^{n+1}}.
\]

Canceling a factor of \(n+1\) from both numerator and denominator yields
\[
\lim_{n \to \infty} \frac{n^n}{(n+1)^{n}}.
\]

Dividing numerator and denominator by \(n^n\) gives
\[
\lim_{n \to \infty} \frac{1}{(1 + \frac{1}{n})^n} = \frac{1}{e}
\]

since \(\lim_{n \to \infty} (1 + \frac{1}{n})^n = e\). Therefore, since \(\frac{1}{e} < 1\), the Ratio Test says that the series converges absolutely.

22. Is the series
\[
\sum_{n=2}^{\infty} \left( \frac{-2n}{n+1} \right)^{5n}
\]
absolutely convergent, conditionally convergent, or divergent?

**Answer:** Using the Root Test,
\[
\lim_{n \to \infty} \sqrt[n]{\left| \frac{-2n}{n+1} \right|^{5n}} = \lim_{n \to \infty} \sqrt[n]{\frac{2n}{n+1}^{5n}} = \lim_{n \to \infty} \left( \frac{2n}{n+1} \right)^{5n}.
\]

Since \(\lim_{n \to \infty} \frac{2n}{n+1} = 2\), the above limit is equal to 32, which is certainly > 1. Therefore, by the Root Test, the series diverges.

24. Is the series
\[
\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}
\]
absolutely convergent, conditionally convergent, or divergent?

**Answer:** Using the Root Test,
\[
\lim_{n \to \infty} \sqrt[n]{\frac{n}{(\ln n)^n}} = \lim_{n \to \infty} \frac{\sqrt[n]{n}}{\ln n} = 0
\]

since \(\lim_{n \to \infty} \sqrt[n]{n} = 1\) and \(\lim_{n \to \infty} \ln n = \infty\). Therefore, the Root Test says that the series converges absolutely.

30. The series \(\sum a_n\) is defined by the equations
\[
a_1 = 1 \quad a_{n+1} = \frac{2 + \cos n}{\sqrt{n}} a_n.
\]
Determine whether $\sum a_n$ converges or diverges.

**Answer:** Using the Ratio Test,

$$
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{2 + \cos n}{\sqrt{n} a_n} \right| = \lim_{n \to \infty} \frac{|2 + \cos n|}{\sqrt{n}}. \quad (*)
$$

Since $|\cos n| \leq 1$, we know that

$$
|2 + \cos n| \leq 2 + |\cos n| \leq 3,
$$

so the above limit is bounded above by

$$
\lim_{n \to \infty} \frac{3}{\sqrt{n}} = 0.
$$

On the other hand, since the limit in $(*)$ has to be non-negative, we see that it is actually equal to 0. Hence, since $0 < 1$, the Ratio Test says that the series converges absolutely.