Math 115 HW #3 Solutions

From §12.2

20. Determine whether the geometric series

\[ \sum_{n=1}^{\infty} e^{n} \left( \frac{3}{n-1} \right) \]

is convergent or divergent. If it is convergent, find its sum.

**Answer:** I can re-write the terms as

\[ \frac{e^{n}}{3^{n-1}} = \frac{e^{n-1}}{3^{n-1}} = e \left( \frac{e}{3} \right)^{n-1}. \]

Therefore, the series

\[ \sum_{n=1}^{\infty} e^{n} \left( \frac{3}{n-1} \right) = e \sum_{n=1}^{\infty} \left( \frac{e}{3} \right)^{n-1} = e \sum_{n=0}^{\infty} \left( \frac{e}{3} \right)^{n}, \]

where the second equality comes from shifting the index by one. Since \( \frac{e}{3} < 1 \), we know that the geometric series

\[ \sum_{n=0}^{\infty} \left( \frac{e}{3} \right)^{n} = \frac{1}{1-\frac{e}{3}} = \frac{3}{3-e}. \]

Therefore, the given series converges and the sum is given by

\[ \sum_{n=1}^{\infty} e^{n} \left( \frac{3}{n-1} \right) = e \sum_{n=0}^{\infty} \left( \frac{e}{3} \right)^{n} = e \frac{3}{3-e} = \frac{3e}{3-e}. \]

24. Determine whether the series

\[ \sum_{k=1}^{\infty} \frac{k(k + 2)}{(k + 3)^{2}} \]

is convergent or divergent. If it is convergent, find its sum.

**Answer:** This series diverges. To see this, I will show that the terms in the sequence do not go to zero:

\[ \lim_{k \to \infty} \frac{k(k + 2)}{(k + 3)^{2}} = \lim_{k \to \infty} \frac{k^{2} + 2k}{k^{2} + 6k + 9}. \]

Dividing numerator and denominator by \( k^{2} \) yields

\[ \lim_{k \to \infty} \frac{1}{k^{2}} \left( \frac{1 + \frac{2}{k}}{1 + \frac{6}{k} + \frac{9}{k^{2}}} \right) = \lim_{k \to \infty} \frac{1 + \frac{2}{k}}{1 + \frac{6}{k} + \frac{9}{k^{2}}} = 1. \]

Therefore, using the n\(^{th}\) term test (a.k.a. Test for Divergence), the series diverges.
38. Determine whether the series
\[ \sum_{n=1}^{\infty} \ln \frac{n}{n+1} \]
is convergent or divergent by expressing \( s_n \) as a telescoping sum. If it is convergent, find its sum.

**Answer:** We can re-write the terms in the series as
\[ \ln \left( \frac{n}{n+1} \right) = \ln(n) - \ln(n+1). \]

Therefore, the partial sum
\[ s_n = a_1 + a_2 + a_3 + \ldots + a_n \]
\[ = (\ln(1) - \ln(2)) + (\ln(2) - \ln(3)) + (\ln(3) - \ln(4)) \ldots + (\ln(n) - \ln(n+1)) \]
\[ = \ln(1) + (-\ln(2) + \ln(2)) + (-\ln(3) + \ln(3)) + \ldots + (-\ln(n) + \ln(n)) - \ln(n + 1) \]
\[ = \ln(1) - \ln(n + 1) \]
\[ = -\ln(n + 1). \]

Therefore,
\[ \sum_{n=1}^{\infty} \ln \left( \frac{n}{n+1} \right) = \lim_{n \to \infty} s_n = \lim_{n \to \infty} (-\ln(n + 1)) = -\infty, \]
so the given series diverges.

48. Find the values of \( x \) for which the series
\[ \sum_{n=1}^{\infty} (x - 4)^n \]
converges. Find the sum of the series for those values of \( x \).

**Answer:** Notice that this is a geometric series, so the series converges when \( |x - 4| < 1 \), meaning that
\[ 3 < x < 5. \]

Moreover, for such values of \( x \), the series converges to
\[ \frac{1}{1 - (x - 4)} = \frac{1}{4 - x}. \]

70. If \( \sum a_n \) and \( \sum b_n \) are both divergent, is \( \sum (a_n + b_n) \) necessarily divergent?

**Answer:** No. Let \( a_n = 1 \) for all \( n \) and let \( b_n = -1 \). Then
\[ \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 1 \text{ diverges} \]

and
\[ \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (-1) \text{ diverges} \]
However, 
\[ \sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} (1 + (-1)) = \sum_{n=1}^{\infty} 0 = 0 \]
certainly converges.

From §12.3

16. Determine whether the series 
\[ \sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1} \]
is convergent or divergent.

**Answer:** If we let \( f(x) = \frac{x^2}{x^3 + 1} \), then the terms of the series and the function \( f \) satisfy the hypotheses of the Integral Test, so the series will converge if and only if 
\[ \int_{1}^{\infty} f(x)\,dx = \int_{1}^{\infty} \frac{x^2}{x^3 + 1}\,dx \]
is finite.

Letting \( u = x^3 + 1 \), we have that \( du = 3x^2\,dx \), so I can re-write the above integral as
\[ \frac{1}{3} \int_{u=2}^{\infty} \frac{du}{u} = \frac{1}{3} \left[ \ln |u| \right]_{2}^{\infty}, \]
which diverges since \( \ln(u) \to \infty \) as \( u \to \infty \). Therefore, the series diverges by the Integral Test.

22. Determine whether the series 
\[ \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \]
is convergent or divergent.

**Answer:** If we let \( f(x) = \frac{1}{x(\ln x)^2} \), then the terms of the series and the function \( f \) satisfy the hypotheses of the Integral Test, so the series will converge if and only if 
\[ \int_{2}^{\infty} f(x)\,dx = \int_{2}^{\infty} \frac{1}{x(\ln x)^2}\,dx \]
is finite.

Letting \( u = \ln x \), we have that \( du = \frac{1}{x}\,dx \), so I can re-write the above integral as
\[ \int_{u=\ln 2}^{\infty} \frac{du}{u^2} = - \left. u^{-1} \right|_{u=\ln 2}^{\infty} = \frac{1}{\ln^2}, \]
which is finite. Therefore, the series converges by the Integral Test.
30. Find the values of \( p \) for which the series

\[
\sum_{n=1}^{\infty} \frac{\ln n}{n^p}
\]

is convergent.

**Answer:** When \( p \leq 0 \) the terms in the series do not go to zero, so the series will diverge. When \( p > 0 \), the function \( f(x) = \frac{\ln x}{x^p} \) and the series satisfy the hypotheses of the Integral Test, so the series will converge if and only if

\[
\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{\ln x}{x^p}dx
\]

is finite.

When \( p = 1 \), I can write the above integral as

\[
\int_1^{\infty} \frac{\ln x}{x}dx.
\]

Letting \( u = \ln x \), we have \( du = \frac{1}{x}dx \), so this is equal to

\[
\int_{u=0}^{\infty} udu = \left[ \frac{u^2}{2} \right]_{0}^{\infty},
\]

which is infinite.

When \( p \neq 1 \), I will use integration by parts to evaluate the integral in (*) . Letting \( u = \ln x \) and \( dv = \frac{dx}{x^p} \), I have that

\[
\begin{align*}
  u &= \ln x & dv &= x^{-p}dx \\
  du &= \frac{1}{x}dx & v &= \frac{x^{1-p}}{1-p},
\end{align*}
\]

so the anti-derivative is given by

\[
\int \frac{\ln x}{x^p}dx = (\ln x) \frac{x^{1-p}}{1-p} - \frac{1}{1-p} \int \frac{dx}{x^p}
\]

\[
= (\ln x) \frac{x^{1-p}}{1-p} - \frac{1}{1-p} \frac{x^{1-p}}{1-p} 1-p
\]

\[
= x^{1-p} (1-p) \ln x - 1
\]

\[
\frac{(1-p)^2}{(1-p)^2}.
\]

Hence,

\[
\int_1^{\infty} \frac{\ln x}{x^p}dx = x^{1-p} (1-p) \ln x - 1 \left[ \frac{(1-p)^2}{(1-p)^2} \right]_1^{\infty},
\]

which is finite only when \( 1 - p < 0 \).

Therefore, the series converges when \( p > 1 \).
34. Find the sum of the series \( \sum_{n=1}^{\infty} \frac{1}{n^5} \) correct to three decimal places.

**Answer:** If we estimate the sum by the \( n \)th partial sum \( s_n \), then we know that the remainder \( R_n \) is bounded by

\[
\int_{n+1}^{\infty} \frac{1}{x^5} \, dx \leq R_n \leq \int_{n}^{\infty} \frac{1}{x^5} \, dx.
\]

This means that

\[
R_n \leq \int_{n}^{\infty} \frac{1}{x^5} \, dx = -\frac{1}{4} \left. \frac{1}{x^4} \right|_{n}^{\infty} = \frac{1}{4n^4},
\]

so the estimate will be accurate to 3 decimal places when this expression is less than 0.001. In other words, we want to know for what \( n \) is it true that

\[
\frac{1}{4n^4} < \frac{1}{1000}.
\]

Solving for \( n \), we get that

\[
n > \sqrt[4]{250} \approx 3.98.
\]

So letting \( n = 4 \), we have that

\[
s_4 = \frac{1}{1^5} + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} = 1 + \frac{1}{32} + \frac{1}{243} + \frac{1}{1024} \approx 1.036
\]

is an estimate of the sum of the series that is correct to three decimal places.

**From §12.4**

18. Determine whether the series

\[
\sum_{n=1}^{\infty} \frac{1}{2n+3}
\]

converges or diverges.

**Answer:** Use the Limit Comparison Test to compare this series to \( \sum \frac{1}{n} \). We see that

\[
\lim_{n \to \infty} \frac{\frac{1}{2n+3}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n}{2n+3} = \frac{1}{2}
\]

Therefore, since \( \sum \frac{1}{n} \) diverges, the Limit Comparison Test tells us that the series \( \sum \frac{1}{2n+3} \) also diverges.

26. Determine whether the series

\[
\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7} + n^2}
\]

converges or diverges.

**Answer:** Use the Limit Comparison Test to compare this series to \( \sum \frac{1}{n^{4/3}} \). We see that

\[
\lim_{n \to \infty} \frac{\frac{n+5}{\sqrt[3]{n^7} + n^2}}{\frac{1}{n^{4/3}}} = \lim_{n \to \infty} \frac{(n+5)n^{4/3}}{\sqrt[3]{n^7} + n^2} = \lim_{n \to \infty} \frac{n^{7/3} + 5n^{4/3}}{\sqrt[3]{n^7} + n^2}.
\]
Therefore, dividing both numerator and denominator by \( n^{7/3} \), we see that this limit is equal to

\[
\lim_{n \to \infty} \frac{1}{n^{7/3}} \left( \frac{n^{7/3} + 5n^{4/3}}{n^{7/3} \sqrt{n} + n^2} \right) = \lim_{n \to \infty} \frac{1 + \frac{5}{n}}{\sqrt[n]{1 + \frac{1}{n^5}}} = 1.
\]

Therefore, since \( \sum \frac{1}{n^{7/3}} \) converges, the Limit Comparison Test tells us that the given series also converges.

32. Determine whether the series

\[
\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}
\]

converges or diverges.

**Answer:** Use the Limit Comparison Test to compare this series to \( \sum \frac{1}{n} \). We see that

\[
\lim_{n \to \infty} \frac{1}{n^{1+1/n}} = \lim_{n \to \infty} \frac{n}{n^{1+1/n}} = \lim_{n \to \infty} \frac{n}{n^{1/n} \cdot n^{1/n}} = \lim_{n \to \infty} \frac{1}{n^{1/n}} = 1
\]

since \( \lim_{n \to \infty} \sqrt[n]{n} = 1 \). Therefore, since \( \sum \frac{1}{n} \) diverges, the Limit Comparison test tells us that \( \sum \frac{1}{n^{1+1/n}} \) also diverges.