Math 115 HW #2 Solutions

1. In the special theory of relativity, the mass of a particle with velocity \( v \) is given by

\[
m = \frac{m_0}{\sqrt{1 - v^2/c^2}}
\]

where \( m_0 \) is the mass of the particle at rest and \( c \) is the speed of light. What happens as \( v \to c^- \)?

**Answer:** As \( v \to c^- \), the fraction \( \frac{v^2}{c^2} \) gets closer and closer to 1 (though it is always less than 1 since \( v \) is approaching \( c \) from the left). Hence, the quantity \( 1 - \frac{v^2}{c^2} \) is approaching zero. Therefore, as \( v \to c^- \), the mass:

\[
m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \to +\infty.
\]

2. Evaluate

\[
limit_{t \to 0} \left( \frac{1}{t} - \frac{1}{t^2 + t} \right).
\]

**Answer:** Notice that \( t^2 + t = t(t + 1) \). Therefore,

\[
\frac{1}{t} - \frac{1}{t^2 + t} = \frac{t + 1}{t(t + 1)} - \frac{1}{t(t + 1)} = \frac{t}{t(t + 1)} = \frac{1}{t + 1}.
\]

Therefore,

\[
lim_{t \to 0} \left( \frac{1}{t} - \frac{1}{t^2 + t} \right) = \lim_{t \to 0} \frac{1}{t + 1} = 1.
\]

3. Evaluate

\[
limit_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}}.
\]

**Answer:** Using L'Hôpital's Rule,

\[
limit_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \to \infty} \frac{1}{\frac{1}{2} \cdot 2x} = \lim_{x \to \infty} \frac{1}{\sqrt{x^2 + 1}} = \lim_{x \to \infty} \frac{1}{\sqrt{1 + 1/x^2}} = 1.
\]

provided that limit exists. If we knew the limit existed, then it would have to be 1, since the left and right sides are reciprocals of each other. However, we don’t know, *a priori*, that the limit exists, so we must use another method.

Let’s divide both the numerator and denominator by \( x \):

\[
limit_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\sqrt{1 + 1/x^2}} = \lim_{x \to \infty} \frac{1}{\sqrt{1 + 1/x^2}} = \lim_{x \to \infty} \frac{1}{\sqrt{1 + 1/x^2}} = 1.
\]
Stewart §12.1

22. Let 
\[ a_n = \frac{3^{n+2}}{5^n} . \]
Determine whether the sequence \((a_n)\) converges or diverges. If it converges, find the limit. 
**Answer:** We can re-write the terms in the sequence as 
\[ a_n = \frac{3^{n+2}}{5^n} = 5 \cdot \left( \frac{3}{5} \right)^n . \]
Since \(\left( \frac{3}{5} \right)^n \to 0\) as \(n \to \infty\), we see that 
\[ \lim_{n \to \infty} \frac{3^{n+2}}{5^n} = 5 \lim_{n \to \infty} \left( \frac{3}{5} \right)^n = 0 . \]

28. Let 
\[ a_n = \cos\left(\frac{2}{n}\right) . \]
Determine whether the sequence \((a_n)\) converges or diverges. If it converges, find the limit. 
**Answer:** Consider the function 
\[ f(x) = \cos\left(\frac{2}{x}\right) . \]
Then, using L'Hôpital's Rule, 
\[ \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \cos\left(\frac{2}{x}\right) = \cos\left(\lim_{x \to \infty} \frac{2}{x}\right) = \cos(0) = 1 . \]
Therefore, 
\[ \lim_{n \to \infty} \cos\left(\frac{2}{n}\right) = \lim_{x \to \infty} f(x) = 1 . \]

34. Let 
\[ a_n = n \cos n\pi . \]
Determine whether the sequence \((a_n)\) converges or diverges. If it converges, find the limit. 
**Answer:** Notice that \(\cos n\pi = (-1)^n\), so we can re-write the terms as 
\[ a_n = n \cos n\pi = n(-1)^n . \]
The sequence is unbounded, so it diverges.

42. Let 
\[ a_n = \frac{(\ln n)^2}{n} . \]
Determine whether the sequence \((a_n)\) converges or diverges. If it converges, find the limit. 
**Answer:** Consider the function 
\[ f(x) = \frac{(\ln x)^2}{x} . \] Then, using L'Hôpital's Rule, 
\[ \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{(\ln x)^2}{x} = \lim_{x \to \infty} \frac{2 \ln x \frac{1}{x}}{1} = \lim_{x \to \infty} \frac{2 \ln x}{x} . \]
Using L'Hôpital’s Rule again, this limit is equal to

\[
\lim_{x \to \infty} \frac{2^{1/x}}{1} = \lim_{x \to \infty} \frac{2}{x} = 0.
\]

Therefore,

\[
\lim_{n \to \infty} \frac{(\ln n)^2}{n} = \lim_{x \to \infty} \frac{(\ln x)^2}{x} = 0.
\]

54. (a) Determine whether the sequence defined as follows is convergent or divergent:

\[
a_1 = 1 \quad a_{n+1} = 4 - a_n \quad \text{for } n \geq 1.
\]

Answer: Writing down the first few terms of the sequence, we see that

\[
a_1 = 1, \quad a_2 = 3, \quad a_3 = 1, \quad a_4 = 3, \quad a_5 = 1, \ldots
\]

The terms oscillate between 1 and 3, so the sequence cannot converge.

(b) What happens if the first term is \(a_1 = 2\)?

Answer: If \(a_1 = 2\), then the sequence looks like

\[
a_1 = 2, \quad a_2 = 2, \quad a_3 = 2, \quad a_4 = 2, \quad a_5 = 2, \ldots
\]

which is as convergent as a sequence could possibly be.

58(b). A sequence \((a_n)\) is defined by \(a_1 = 1\) and \(a_{n+1} = 1/(1 + a_n)\) for \(n \geq 1\). Assuming that \((a_n)\) is convergent, find its limit.

Answer: Suppose \(\lim_{n \to \infty} a_n = L\). Then we have that

\[
L = \lim_{n \to \infty} a_n = \lim_{n \to \infty} a_{n+1} \quad \text{by shifting the index}
\]

\[
= \lim_{n \to \infty} \frac{1}{1 + a_n} \quad \text{by definition of } a_{n+1}
\]

\[
= \frac{1}{1+L} \quad \text{by taking the limit.}
\]

Therefore,

\[
L = \frac{1}{1+L},
\]

meaning that, if we multiply both sides by \(1 + L\),

\[
1 = L(1 + L) = L + L^2.
\]

Equivalently,

\[
0 = L^2 + L - 1,
\]

so \(L = \frac{-1 + \sqrt{5}}{2}\). But since all the terms in the sequence are positive, \(L\) cannot be negative, so

\[
L = \frac{-1 + \sqrt{5}}{2}.
\]
60. Determine whether the sequence
\[ a_n = (-2)^{n+1} \]
is increasing, decreasing, or not monotonic. Is the sequence bounded?
**Answer:** The first few terms of the sequence are
\[-2, 4, -8, 16, -32, \ldots \]
The sequence is neither monotonic nor bounded.

62. Determine whether the sequence
\[ a_n = \frac{2n - 3}{3n + 4} \]
is increasing, decreasing, or not monotonic. Is the sequence bounded?
**Answer:** Let \( f(x) = \frac{2x - 3}{3x + 4} \). Then
\[ f'(x) = \frac{2(3x + 4) - 3(2x - 3)}{(3x + 4)^2} = \frac{6x + 8 - (6x - 9)}{(3x + 4)^2} = \frac{17}{(3x + 4)^2} > 0, \]
so \( f(x) \) is an increasing function. Therefore, since \( f(n) = a_n \), the sequence \((a_n)\) is increasing as well.

Since the sequence is increasing, each term \( a_n > a_1 = \frac{-1}{7} \). On the other hand, each term
\[ a_n = \frac{2n - 3}{3n + 4} < \frac{2n - 3}{3n} < \frac{2n}{3n} = \frac{2}{3}, \]
Therefore, the terms are trapped:
\[ -\frac{1}{7} < a_n < \frac{2}{3} \text{ for all } n, \]
so the sequence is bounded.

70. Show that the sequence defined by
\[ a_1 = 2 \quad a_{n+1} = \frac{1}{3 - a_n} \]
satisfies \( 0 < a_n \leq 2 \) and is decreasing. Deduce that the sequence is convergent and find its limit.
**Answer:** First, we prove by induction that \( 0 < a_n \leq 2 \) for all \( n \).

0: Clearly, \( 0 < a_1 \leq 2 \) since \( a_1 = 2 \).
1: Assume \( 0 < a_n \leq 2 \).
2: Then, using that assumption,
\[ a_{n+1} = \frac{1}{3 - a_n} > \frac{1}{3 - 0} = \frac{1}{3} > 0. \]
Also,
\[ a_{n+1} = \frac{1}{3 - a_n} \leq \frac{1}{3 - 2} = 1 < 2, \]
so we see that the assumption that \( 0 < a_n \leq 2 \) implies that
\[ 0 < a_{n+1} < 2. \]
3: Therefore, by induction, \(0 < a_n \leq 2\) for all \(n\).

Next, we show that \(a_{n+1} \leq a_n\) for all \(n\), again using induction.

0: Clearly, \(a_2 \leq a_1\) since \(a_2 = 1\) and \(a_1 = 2\).

1: Assume \(a_{n+1} \leq a_n\).

2: Then, using that assumption,
\[
a_{n+2} = \frac{1}{3-a_{n+1}} \leq \frac{1}{3-a_n} = a_{n+1}.
\]

3: Therefore, by induction, \(a_{n+1} \leq a_n\) for all \(n\).

We’ve shown that the sequence \((a_n)\) is bounded and decreasing, so the Monotone Convergence Property implies that it converges. Call the limit of the sequence \(L\). Then
\[
L = \lim_{n \to \infty} a_n = \lim_{n \to \infty} a_{n+1} \quad \text{by shifting the index}
\]
\[
= \lim_{n \to \infty} \frac{1}{3 - a_n} \quad \text{by definition of } a_{n+1}
\]
\[
= \frac{1}{3 - L} \quad \text{by taking the limit.}
\]

Therefore,
\[
L(3 - L) = 1,
\]
or, equivalently,
\[
L^2 - 3L + 1 = 0.
\]

Possible solutions to this equation are \(L = \frac{3 \pm \sqrt{5}}{2}\), but, since all the terms of the sequence are between 0 and 2, \(L\) must be between 0 and 2 as well. Therefore,
\[
L = \frac{3 - \sqrt{5}}{2}.
\]