Math 115 HW #10 Solutions

1. Suppose $y_1(t)$ and $y_2(t)$ are both solutions of the differential equation

\[ P(t)y'' + Q(t)y' + R(t)y = 0. \]

Show that, for any constants $C_1$ and $C_2$, the function

\[ C_1y_1(t) + C_2y_2(t) \]

is also a solution of this differential equation.

**Proof.** Let $y = C_1y_1 + C_2y_2$. Then

\[ y' = C_1y_1' + C_2y_2' \]

and

\[ y'' = C_1y_1'' + C_2y_2''. \]

Therefore,

\[ P(t)y'' + Q(t)y' + R(t)y = P(t)(C_1y_1'' + C_2y_2'') + Q(t)(C_1y_1' + C_2y_2') + R(t)(C_1y_1 + C_2y_2) \]

\[ = (P(t)C_1y_1'' + Q(t)C_1y_1' + R(t)C_1y_1) + (P(t)C_2y_2'' + Q(t)C_2y_2' + R(t)C_2y_2) \]

\[ = C_1(P(t)y_1'' + Q(t)y_1' + R(t)y_1) + C_2(P(t)y_2'' + Q(t)y_2' + R(t)y_2). \]

However, both terms in the bottom line are zero, for the following reason: since $y_1$ is a solution of the given differential equation,

\[ P(t)y_1'' + Q(t)y_1' + R(t)y_1 = 0; \]

likewise, since $y_2$ is a solution,

\[ P(t)y_2'' + Q(t)y_2' + R(t)y_2 = 0. \]

Therefore, we see that $y = C_1y_1 + C_2y_2$ is indeed a solution of the given differential equation for any constants $C_1$ and $C_2$.

2. Solve the differential equation

\[ 6y'' - 7y' - 12y = 0. \]

**Answer:** The characteristic equation is

\[ 6r^2 - 7r - 12 = 0; \]

solutions of this equation are:

\[ r = \frac{7 \pm \sqrt{(-7)^2 - 4(6)(-12)}}{2(6)} = \frac{7 \pm \sqrt{49 + 288}}{12} = \frac{7 \pm \sqrt{337}}{12}. \]

Therefore, solutions of the given differential equation are of the form

\[ y = C_1e^{\frac{7+\sqrt{337}}{12}t} + C_2e^{\frac{7-\sqrt{337}}{12}t}. \]
3. Solve the initial-value problem

$$2y'' + 6y' + 17y = 0, \quad y(0) = 1, y'(0) = 5.$$ 

**Answer:** The characteristic equation is

$$2r^2 + 6r + 17 = 0;$$

solutions are

$$r = \frac{-6 \pm \sqrt{6^2 - 4(2)(17)}}{2(2)} = \frac{-6 \pm \sqrt{36 - 136}}{4} = -\frac{3}{2} \pm \frac{5}{2}i.$$

Therefore, solutions of the given differential equation are of the form

$$y = C_1 e^{-\frac{3}{2}t} \cos \left(\frac{5}{2}t\right) + C_2 e^{-\frac{3}{2}t} \sin \left(\frac{5}{2}t\right).$$

Plugging in \(t = 0\), we have that

$$1 = y(0) = C_1 e^0 \cos(0) + C_2 e^0 \sin(0) = C_1,$$

so \(C_1 = 1\) and

$$y = e^{-\frac{3}{2}t} \cos \left(\frac{5}{2}t\right) + C_2 e^{-\frac{3}{2}t} \sin \left(\frac{5}{2}t\right).$$

Hence,

$$y' = -\frac{3}{2} e^{-\frac{3}{2}t} \cos \left(\frac{5}{2}t\right) - \frac{5}{2} e^{-\frac{3}{2}t} \sin \left(\frac{5}{2}t\right) - \frac{3}{2} C_2 e^{-\frac{3}{2}t} \sin \left(\frac{5}{2}t\right) + \frac{5}{2} C_2 e^{-\frac{3}{2}t} \cos \left(\frac{5}{2}t\right).$$

Therefore, plugging in \(t = 0\) yields

$$5 = y'(0) = \left(\frac{5}{2} C_2 - \frac{3}{2}\right) e^0 \cos(0) - \left(\frac{5}{2} + \frac{3}{2} C_2\right) e^0 \sin(0) = \frac{5}{2} C_2 - \frac{3}{2}.$$ 

Therefore,

$$\frac{5}{2} C_2 = 5 + \frac{3}{2} = \frac{13}{2},$$

so

$$C_2 = \frac{2 \cdot 13}{5 \cdot 2} = \frac{13}{5}.$$ 

Thus, we conclude that

$$y = e^{-\frac{3}{2}t} \cos \left(\frac{5}{2}t\right) + \frac{13}{5} e^{-\frac{3}{2}t} \sin \left(\frac{5}{2}t\right).$$
4. A spring-mass-dashpot system (like the door-closing mechanism in many doors) can be modeled by the differential equation

\[ m x'' + cx' + kx = 0 \]

where \( x \) is the displacement of the object, \( m \) is the mass of the object, \( c \) is the damping constant for the dashpot, and \( k \) is the spring constant. Suppose we have such a system with a mass \( m = 20 \) kg, a spring with \( k = 5 \), and a dashpot whose damping constant \( c \) we can adjust. What value of \( c \) should we pick to get critical damping?

**Answer:** Critical damping occurs when there is a single root (of multiplicity 2) of the characteristic equation. Plugging in the values for \( m \) and \( k \) we have the differential equation

\[ 20 x'' + cx' + 5x = 0, \]

so the characteristic equation is

\[ 20r^2 + cr + 5 = 0. \]

Solutions are of the form

\[ r = \frac{-c \pm \sqrt{c^2 - 4(20)(5)}}{2(20)} = \frac{-c \pm \sqrt{c^2 - 400}}{40}. \]

There is a single root of multiplicity 2 when the discriminant \( c^2 - 400 = 0 \), meaning that

\[ c = \pm 20. \]

The value \( c = -20 \) is physically meaningless, so we should pick \( c = 20 \) to get critical damping.

5. Solve the differential equation

\[ y'' - y' - 6y = e^{2x}. \]

**Answer:** First, we solve the homogeneous equation

\[ y'' - y' - 6y = 0. \]

This has characteristic equation

\[ r^2 - r - 6 = 0; \]

The left side factors as \((r - 3)(r + 2)\), so the roots are \( r_1 = 3 \) and \( r_2 = -2 \). Hence, the complementary solution (i.e. solution to the homogeneous equation) is

\[ y_c = C_1 e^{3x} + C_2 e^{-2x}. \]

Now, we need to find a particular solutions to the given equation

\[ y'' - y' - 6y = e^{2x}. \]

Using the method of undetermined coefficients, guess that

\[ y_p = Ae^{2x}. \]
Then
\[ y'_p = 2Ae^{2x}, \]
and
\[ y''_p = 4Ae^{2x}. \]
Therefore, if \( y_p \) really is a solution, we should have that
\[ e^{2x} = y''_p - y'_p - 6y_p = 4Ae^{2x} - 2Ae^{2x} - 6(Ae^{2x}) = -4Ae^{2x}. \]
Therefore, it must be the case that
\[ 1 = -4A, \]
so \( A = -\frac{1}{4} \) and
\[ y_p = -\frac{1}{4}e^{2x}. \]
Thus, the general solution of the given non-homogeneous equation is
\[ y = y_c + y_p = C_1e^{3x} + C_2e^{-2x} - \frac{1}{4}e^{2x}. \]

6. Solve the differential equation
\[ y'' - 4y' + 4y = e^{2x}. \]

**Answer:** First, solve the homogeneous equation
\[ y'' - 4y' + 4y = 0. \]
This has characteristic equation
\[ r^2 - 4r + 4 = 0 \]
and the left side factors as \((r-2)^2\), so the single solution (of multiplicity 2) is \( r = 2 \). Therefore, the complementary solution is
\[ y_c = C_1e^{2x} + C_2xe^{2x}. \]
Now, to find a particular solution to the given equation, we would like to guess that \( y_p \) is \( e^{2x} \). However, this is already a solution to the homogeneous equation, so it can’t be a particular solution. Multiplying by \( x \) yields \( xe^{2x} \), which is also a solution to the homogeneous equation. Therefore, we need to multiply by \( x \) again and guess
\[ y_p = Ax^2e^{2x}. \]
Then
\[ y'_p = 2Axe^{2x} + 2Ax^2e^{2x} \]
and
\[ y''_p = 2Ae^{2x} + 4Axe^{2x} + 4Axe^{2x} + 4Ax^2e^{2x} = 2Ae^{2x} + 8Axe^{2x} + 4Ax^2e^{2x}. \]
Therefore, since $y_p$ is a solution of the equation,

\[
e^{2x} = y_p'' - 4y'_p + 4y_p
\]

\[
= (2Ae^{2x} + 8Axe^{2x} + 4Ax^2e^{2x}) - 4(2Axe^{2x} + 2Ax^2e^{2x}) + 4Ax^2e^{2x}
\]

\[
= (4A - 8A + 4A)x^2e^{2x} + (8A - 8A)xe^{2x} + 2Ae^{2x}
\]

\[
= 2Ae^{2x}.
\]

Therefore, $2A = 1$ and so $A = 1/2$. Hence,

\[
y_p = \frac{1}{2}x^2e^{2x}
\]

and so the solution of the differential equation is

\[
y = y_c + y_p = C_1e^{2x} + C_2xe^{2x} + \frac{1}{2}x^2e^{2x}.
\]

7. Solve the initial-value problem

\[
y'' + 9y = \cos 3x + \sin 3x, \quad y(0) = 2, \quad y'(0) = 1.
\]

**Answer:** First, solve the homogeneous equation

\[
y'' + 9y = 0.
\]

This equation has characteristic equation

\[
r^2 + 9 = 0,
\]

which has solutions $r = \pm 3i$. Therefore, the complementary solution is

\[
y_c = C_1e^{0x}\cos 3x + C_2e^{0x}\sin 3x
\]

\[
= C_1\cos 3x + C_2\sin 3x.
\]

Now, we would like to guess that the particular solution is $A\cos 3x + B\sin 3x$, but both $\cos 3x$ and $\sin 3x$ are solutions to the homogeneous equation. Hence, we multiply by $x$ and guess that

\[
y_p = Ax\cos 3x + Bx\sin 3x.
\]

Then

\[
y_p' = A\cos 3x - 3Ax\sin 3x + B\sin 3x + 3Bx\cos 3x
\]

\[
= (A + 3Bx)\cos 3x + (B - 3Ax)\sin 3x
\]

and so

\[
y_p'' = -3A\sin 3x + 3B\cos 3x - 9Bx\sin 3x + 3B\cos 3x - 3A\sin 3x - 9Ax\cos 3x
\]

\[
= (6B - 9Ax)\cos 3x - (6A + 9Bx)\sin 3x.
\]
Therefore, since $y_p$ solves the differential equation,

$$
cos 3x + sin 3x = y''_p + 9y_p
$$

$$
= [(6B - 9Ax) \cos 3x - (6A + 9Bx) \sin 3x] + 9[Ax \cos 3x + Bx \sin 3x]
$$

$$
= 6B \cos 3x - 6A \sin 3x
$$

Therefore

$$
1 = 6B, \quad 1 = -6A,
$$

so $B = 1/6$ and $A = -1/6$, meaning that

$$
y_p = -\frac{1}{6}x \cos 3x + \frac{1}{6}x \sin 3x.
$$

Hence,

$$
y = y_c + y_p = C_1 \cos 3x + C_2 \sin 3x - \frac{1}{6}x \cos 3x + \frac{1}{6}x \sin 3x.
$$

Now, plugging in $x = 0$ yields

$$
2 = y(0) = C_1 \cos(0) + C_2 \sin(0) - \frac{1}{6}(0) \cos(0) + \frac{1}{6}(0) \sin(0) = C_1,
$$

so $C_1 = 2$ and

$$
y = 2 \cos 3x + C_2 \sin 3x - \frac{1}{6}x \cos 3x + \frac{1}{6}x \sin 3x.
$$

Our other initial value is $y'(0) = 1$, so we need to find the derivative of $y$:

$$
y' = -6 \sin 3x + 3C_2 \cos 3x - \frac{1}{6} \cos 3x + \frac{1}{2}x \sin 3x + \frac{1}{6} \sin 3x + \frac{1}{2}x \cos 3x
$$

$$
= \left(3C_2 - \frac{1}{6} + \frac{1}{2}x\right) \cos 3x + \left(-6 + \frac{1}{6} + \frac{1}{2}x\right) \sin 3x.
$$

Plugging in $x = 0$ yields

$$
1 = y'(0) = \left(3C_2 - \frac{1}{6} + \frac{1}{2}(0)\right) \cos(0) + \left(-6 + \frac{1}{6} + \frac{1}{2}(0)\right) \sin(0)
$$

$$
= 3C_2 - \frac{1}{6}.
$$

Hence,

$$
3C_2 = \frac{7}{6}
$$

so $C_2 = \frac{7}{18}$.

Therefore, finally, we see that

$$
y = 2 \cos 3x + \frac{7}{18} \sin 3x - \frac{x}{6} \cos 3x + \frac{x}{6} \sin 3x.
$$