Math 115 Exam #3 Practice Problems

1. Solve the initial-value problem \( \frac{dx}{dt} + 2tx = x \), \( x(0) = 5 \). Use your solution to compute \( x(3) \).
   **Answer:** Subtracting \( 2tx \) from both sides and factoring the \( x \) on the right hand side yields the separable equation
   \[
   \frac{dx}{dt} = x(1 - 2t).
   \]
   Hence, we can separate variables and integrate
   \[
   \int \frac{dx}{x} = \int (1 - 2t)dt.
   \]
   Therefore,
   \[
   \ln |x| = t - t^2 + C.
   \]
   Exponentiating both sides yields
   \[
   |x| = e^{t-t^2+C} = Ae^{-t^2}.
   \]
   By allowing \( A \) to be negative, we can eliminate the absolute value signs on the left hand side, so
   \[
   x = Ae^{-t^2}.
   \]
   Plugging in \( t = 0 \) yields
   \[
   5 = Ae^{0-0^2} = A,
   \]
   so we have that
   \[
   x = 5e^{-t^2}.
   \]
   Therefore,
   \[
   x(3) = 5e^{3-3^2} = 5e^{-6} = \frac{5}{e^6}.
   \]

2. Solve the differential equation \( 7yy' = 5x \).
   **Answer:** In this equation the variable are already separated, so we can integrate both sides:
   \[
   \int 7ydy = \int 5xdx.
   \]
   Thus,
   \[
   \frac{7}{2}y^2 = \frac{5}{2}x^2 + C.
   \]
   Therefore,
   \[
   y^2 = \frac{2}{7} \left( \frac{5}{2}x^2 + C \right) = \frac{5}{7}x^2 + C',
   \]
   so
   \[
   y = \pm \sqrt{\frac{5}{7}x^2 + C'}.
   \]

3. Solve the initial-value problem \( y' + y = 2 \), \( y(0) = 1 \).
   **Answer:** This is a linear equation with \( P(x) = 1 \) and \( Q(x) = 2 \). Therefore,
   \[
   \int P(x)dx = \int 1dx = x,
   \]
   so the integrating factor
   \[
   \mu(x) = e^{\int P(x)dx} = e^x.
   \]
In turn, this means that
\[ \int \mu(x)Q(x)dx = \int 2e^x dx = 2e^x + C. \]
Hence,
\[ y = \frac{1}{\mu(x)} \int \mu(x)Q(x)dx = \frac{1}{e^x} (2e^x + C) = 2 + \frac{C}{e^x}. \]
Plugging in \( x = 0 \) yields
\[ 1 = 2 + \frac{C}{e^0} = 2 + C, \]
so \( C = -1 \). Therefore, the solution is
\[ y = 2 - \frac{1}{e^x}. \]
(Note: the given equation is also separable: \( y' = 2 - y \), so could separate and integrate as \( \int \frac{dy}{2-y} = \int dx \)).

4. Solve the initial-value problem \( \frac{dy}{dx} = 1 - y + x^2 - yx^2, \ y(0) = 0. \)

**Answer:** We can factor the right hand side as \( (1 - y)(1 + x^2) \), so this equation is equivalent to the separable equation
\[ \frac{dy}{dx} = (1 - y)(1 + x^2). \]
Separate variables and integrate:
\[ \int \frac{dy}{1 - y} = \int (1 + x^2)dx. \]
Hence,
\[ -\ln |1 - y| = x + \frac{x^3}{3} + C. \]
Multiplying both sides by \(-1\) and exponentiating yields
\[ |1 - y| = e^{-x-x^3/3-C} = Ae^{-x-x^3/3}. \]
Allowing \( A \) to be positive or negative allows us to eliminate the absolute value signs, so
\[ 1 - y = Ae^{-x-x^3/3}. \]
Therefore,
\[ y = 1 - Ae^{-x-x^3/3}. \]
Plugging in \( x = 0 \) yields
\[ 0 = 1 - Ae^{-0-0^3/3} = 1 - A, \]
so \( A = 1 \). Therefore,
\[ y = 1 - e^{-x-x^3/3}. \]

5. Solve the differential equation \( x \frac{dy}{dx} = y^2. \)

**Answer:** This is a separable equation, so we can separate variables and integrate:
\[ \int \frac{dy}{y^2} = \int \frac{dx}{x}. \]
Hence,
\[ -\frac{1}{y} = \ln |x| + C. \]
Solving for \( y \), we see that
\[ y = -\frac{1}{\ln |x| + C}. \]
6. Solve the differential equation \((x^2 + 1) \frac{dy}{dx} = y.\)

**Answer:** This is a separable equation, so we can separate variables and integrate:

\[
\int \frac{dy}{y} = \int \frac{dx}{x^2 + 1}.
\]

Hence,

\[
\ln |y| = \tan^{-1} x + C.
\]

Exponentiating both sides yields,

\[
|y| = e^{\tan^{-1} x + C} = Ae^{\tan^{-1} x}.
\]

Allowing \(A\) to be either positive or negative allows us to eliminate the absolute value signs, so

\[
y = Ae^{\tan^{-1} x}.
\]

7. Solve the initial-value problem \(\frac{dy}{dx} = xy,\ y(1) = 3.\)

**Answer:** This is a separable equation, so we can separate variables and integrate:

\[
\int \frac{dy}{y} = \int xdx.
\]

Thus,

\[
\ln |y| = \frac{x^2}{2} + C.
\]

Exponentiating both sides yields

\[
|y| = e^{\frac{x^2}{2} + C} = Ae^{\frac{x^2}{2}}.
\]

Allowing \(A\) to be either positive or negative lets us get rid of the absolute value signs, so

\[
y = Ae^{\frac{x^2}{2}}.
\]

Plugging in \(x = 1,\) we see that

\[
3 = Ae^{\frac{1^2}{2}} = Ae^{\frac{1}{2}} = A\sqrt{e}.
\]

Therefore, \(A = \frac{3}{\sqrt{e}},\) so the solution is

\[
y = \frac{3}{\sqrt{e}}e^{\frac{x^2}{2}}.
\]

8. In a second-order chemical reaction, the reactant \(A\) is used up in such a way that the amount of it present decreases at a rate proportional to the square of the amount present. Suppose this reaction begins with 50 grams of \(A\) present, and after 10 seconds there are only 25 grams left. How long after the beginning of the reaction will there be only 10 grams left? Will all of the \(A\) disappear in a finite time, or will there always be a little bit present?

**Answer:** Let \(A(t)\) denote the amount of the reactant present after \(t\) seconds. Since the amount present decreases at a rate proportional to the amount present,

\[
\frac{dA}{dt} = kA^2.
\]

This is a separable equation, so we separate variables and integrate:

\[
\int \frac{dA}{A^2} = \int kdt.
\]
Therefore, \[- \frac{1}{A} = kt + C,\]
so, solving for \(A,\)
\[A = - \frac{1}{kt + C}.\]
Plugging in \(t = 0,\) we have that
\[50 = - \frac{1}{k(0) + C} = - \frac{1}{C},\]
so \(C = - \frac{1}{50}.\) In turn, plugging in \(t = 10\) yields
\[25 = - \frac{1}{k(10) - \frac{1}{50}} = - \frac{1}{10k - \frac{1}{50}}.\]
Hence,
\[10k - \frac{1}{50} = - \frac{1}{25},\]
meaning that
\[10k = - \frac{1}{25} + \frac{1}{50} = - \frac{1}{50},\]
so
\[k = - \frac{1}{500}.\]
Therefore, the amount of reactant \(A\) present after \(t\) seconds is
\[A(t) = - \frac{1}{\frac{t}{500} - \frac{1}{50}} = - \frac{1}{\frac{t}{50} + 10} = - \frac{500}{t + 10} = \frac{500}{t + 10}.\]
The time \(t_0\) after which there will only be 10 grams left is determined implicitly by
\[10 = \frac{500}{t_0 + 10},\]
so there will be 10 grams left after \(t_0 = 40\) seconds.
Since \(A(t) = \frac{500}{t + 10}\) and the numerator can never be zero, the amount of \(A\) present is never equal to zero.

9. Market research has shown the price \(p\) and weekly sales \(S(p)\) of a particular product are related by the following differential equation:
\[
\frac{dS}{dp} = - \frac{1}{2} \left( \frac{S}{p + 3} \right).
\]
If sales amount to 100 units when the price is $1 (i.e., \(S(1) = 100\)), what will the weekly sales be if the price is raised to $6?  

**Answer:** The above is a separable equation, so we can separate variables and integrate:
\[
\int \frac{dS}{S} = \int - \frac{1}{2} \frac{dp}{p + 3}.
\]
Therefore,
\[
\ln |S| = - \frac{1}{2} \ln |p + 3| + C.
\]
Since neither sales nor price can be negative, we don’t need the absolute value signs. Hence,
\[
\ln S = - \frac{1}{2} \ln(p + 3) + C = \ln(p + 3)^{-1/2} + C.
\]
Exponentiating both sides yields

\[ S = e^{\ln(p+3)^{-1/2} + C} = Ae^{\ln(p+3)^{-1/2}} = A(p + 3)^{-1/2} = \frac{A}{\sqrt{p+3}}. \]

Plugging in \( p = 1 \) gives

\[ 100 = \frac{A}{\sqrt{1+3}} = \frac{A}{\sqrt{4}} = \frac{A}{2}, \]

so \( A = 200 \). Thus,

\[ S(6) = \frac{200}{6+3} = \frac{200}{9} = 22\frac{2}{9}, \]

so if the price is raised to $6, 22 units will be sold (since presumably it’s not possible to sell \( \frac{2}{9} \) of a unit).

10. Solve the initial-value problem \( \frac{dy}{dx} = \frac{e^{2x}}{6y^5}, \ y(0) = 1 \).

**Answer:** This is a separable equation, so separate variables and integrate:

\[ \int 6y^5 dy = \int e^{2x} dx. \]

Therefore,

\[ y^6 = \frac{1}{2} e^{2x} + C, \]

so

\[ y = \pm \sqrt[6]{\frac{1}{2} e^{2x} + C}. \]

Plugging in \( x = 0 \) gives

\[ 1 = \pm \sqrt[6]{\frac{1}{2} e^{2(0)} + C} = \pm \sqrt[6]{\frac{1}{2} + C}, \]

so \( C = \frac{1}{2} \). Therefore,

\[ y = \pm \sqrt[6]{\frac{1}{2} e^{2x} + \frac{1}{2}}. \]

11. If \( y(x) \) satisfies the differential equation \( \frac{dy}{dx} = e^{2x-y} \) and \( y(0) = 1 \), then what is \( y(1/2) \)?

**Answer:** We can write this equation as

\[ \frac{dy}{dx} = e^{2x} e^{-y}, \]

which is separable. Separate variables and integrate:

\[ \int e^y dy = \int e^{2x} dx. \]

Therefore,

\[ e^y = \frac{1}{2} e^{2x} + C, \]

so

\[ y = \ln \left( \frac{1}{2} e^{2x} + C \right). \]

Plugging in \( x = 0 \) gives

\[ 1 = \ln \left( \frac{1}{2} e^{2(0)} + C \right) = \ln \left( \frac{1}{2} + C \right), \]
so \( C = e - 1/2 \). Therefore,
\[
y(1/2) = \ln \left( \frac{1}{2} e^{2(1/2)} + e - \frac{1}{2} \right) = \ln \left( \frac{1}{2} + e - \frac{1}{2} \right) = \ln e = 1.
\]

12. Solve the differential equation \( x^2 y' - y = 2x e^{-1/x} \).
**Answer:** Divide everything by \( x^2 \) to get the linear equation in standard form:
\[
y' - \frac{1}{x^2} y = 2x e^{-1/x}.
\]
Here \( P(x) = -\frac{1}{x^2} \) and \( Q(x) = 2x e^{-1/x} \), so
\[
\int P(x) \, dx = \int -\frac{1}{x^2} \, dx = \frac{1}{x}.
\]
Thus,
\[
\mu(x) = e^{\int P(x) \, dx} = e^{1/x}.
\]
In turn, this means
\[
\int \mu(x) Q(x) \, dx = \int e^{1/x} 2x e^{-1/x} \, dx = \int 2x \, dx = x^2 + C.
\]
Therefore,
\[
y = \frac{1}{\mu(x)} \int \mu(x) Q(x) \, dx = \frac{1}{e^{1/x}} \left( x^2 + C \right) = \frac{x^2}{e^{1/x}} + \frac{C}{e^{1/x}}.
\]

13. Solve the initial-value problem \( x y' - y = x \ln x, \ x > 0, \ y(1) = 2 \).
**Answer:** Dividing everything by \( x \) yields the linear equation in standard form:
\[
y' - \frac{1}{x} y = \ln x.
\]
Here \( P(x) = -\frac{1}{x} \) and \( Q(x) = \ln x \), so
\[
\int P(x) \, dx = \int -\frac{1}{x} \, dx = -\ln |x| = \ln \left( \frac{1}{x} \right)
\]
(where we can eliminate the absolute value signs because \( x > 0 \)). Therefore,
\[
\mu(x) = e^{\int P(x) \, dx} = e^{\ln(1/x)} = \frac{1}{x}.
\]
In turn,
\[
\int \mu(x) Q(x) \, dx = \int \frac{1}{x} \ln x \, dx = \frac{(\ln x)^2}{2} + C,
\]
so we have that
\[
y = \frac{1}{\mu(x)} \int \mu(x) Q(x) \, dx = \frac{1}{x} \left( \frac{(\ln x)^2}{2} + C \right) = \frac{x(\ln x)^2}{2} + Cx.
\]
Plugging in \( x = 1 \) yields
\[
2 = \frac{1 \cdot (\ln 1)^2}{2} + C \cdot 1 = C,
\]
so
\[
y = \frac{x(\ln x)^2}{2} + 2x.
\]
14. Solve the initial-value problem \((x^2 + 1) {dy \over dx} + 3x(y - 1) = 0, \ y(0) = 2.\)

**Answer:** Dividing everything by \(x^2 + 1\) yields
\[ y' + \frac{3x}{x^2 + 1} (y - 1) = 0. \]

This isn’t quite in standard form; to get it into standard form, add \(\frac{3x}{x^2 + 1}\) to both sides:
\[ y' + \frac{3x}{x^2 + 1} y = \frac{3x}{x^2 + 1}. \]

Here \(P(x) = Q(x) = \frac{3x}{x^2 + 1}\), so
\[ \int P(x) \, dx = \int \frac{3x}{x^2 + 1} \, dx = \frac{3}{2} \ln(x^2 + 1) = \ln(x^2 + 1)^{3/2}, \]
so
\[ \mu(x) = e^{\int P(x) \, dx} = e^{\ln(x^2+1)^{3/2}} = (x^2 + 1)^{3/2}. \]

Therefore,
\[ \int \mu(x)Q(x) \, dx = \int (x^2 + 1)^{3/2} \frac{3x}{x^2 + 1} \, dx = \int 3x \sqrt{x^2 + 1} \, dx = (x^2 + 1)^{3/2} + C, \]
so the solution is
\[ y = \frac{1}{\mu(x)} \int \mu(x)Q(x) \, dx = \frac{1}{(x^2 + 1)^{3/2}} \left( (x^2 + 1)^{3/2} + C \right) = 1 + \frac{C}{(x^2 + 1)^{3/2}}. \]

Plugging in \(x = 0\) yields
\[ 2 = 1 + \frac{C}{(0^2 + 1)^{3/2}} = 1 + C, \]
so \(C = 1.\) Therefore,
\[ y = 1 + \frac{1}{(x^2 + 1)^{3/2}}. \]

15. Solve the differential equation \(t \ln t {dr \over dt} + r = te^t\) assuming \(t > 1.\)

**Answer:** Dividing everything by \(t \ln t\) yields the linear equation in standard form
\[ r' + \frac{1}{t \ln t} r = \frac{e^t}{\ln t}. \]

Here \(P(t) = \frac{1}{t \ln t}\) and \(Q(t) = \frac{e^t}{\ln t}\), so
\[ \int P(t) \, dt = \int \frac{1}{t \ln t} \, dt = \ln |\ln t| = \ln \ln t \]
since \(t > 1.\) Therefore,
\[ \mu(t) = e^{\int P(t) \, dt} = e^{\ln \ln t} = \ln t \]
and so
\[ \int \mu(t)Q(t) \, dt = \int \ln t \frac{e^t}{\ln t} \, dt = \int e^t \, dt = e^t + C. \]

Hence,
\[ r = \frac{1}{\mu(t)} \int \mu(t)Q(t) \, dt = \frac{1}{\ln t} \left( e^t + C \right) = \frac{e^t}{\ln t} + \frac{C}{\ln t}. \]
16. In the following predator-prey system, determine which of the variables, $x$ or $y$, represents the prey population and which represents the predator population. Do the predators feed only on the prey or do they have additional food sources? Explain.

\[
\frac{dx}{dt} = -0.05x + 0.0001xy \\
\frac{dy}{dt} = 0.1y - 0.005xy
\]

**Answer:** Since the $y$ population will grow exponentially without $x$, whereas $x$ will die off without $y$, $y$ must be the prey population and $x$ must be the predator population.

If the predators had another food source then they wouldn’t go extinct in the absence of the prey $y$; however, it’s clear from the given system of equations that the predators *will* die out in the absence of $y$, so they must not have another food source.