Math 113 HW #12 Solutions

1. Exercise 5.2.18. Express the limit

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{\cos x_i}{x_i} \Delta x$$

as a definite integral on $[\pi, 2\pi]$.

**Answer:** This is simply the definition of the definite integral

$$\int_{\pi}^{2\pi} \frac{\cos x}{x} dx.$$

2. Exercise 5.2.34. The graph of $g$ consists of two straight lines and a semicircle. Use it to evaluate each integral

(a) $\int_{0}^{2} g(x) dx$

**Answer:** Since on $[0, 2]$ the graph of $g(x)$ is just a straight line of slope $-2$ coming down from $y = 4$ to $y = 0$, the area is just the area of the triangle

$$\frac{1}{2} \cdot 2 \cdot 4 = 4.$$

Since this area is above the $x$-axis, definite integral equals the area, so $\int_{0}^{2} g(x) dx = 4$.

(b) $\int_{2}^{6} g(x) dx$

**Answer:** On $[2, 6]$ the graph of $g(x)$ is a semi-circle of radius 2 lying below the $x$-axis. Its area is

$$\frac{1}{2} \pi (2)^2 = 2\pi.$$

Since it lies below the axis, the integral is negative, so

$$\int_{2}^{6} g(x) dx = -2\pi.$$

(c) $\int_{6}^{7} g(x) dx$

**Answer:** Since

$$\int_{0}^{7} g(x) dx = \int_{0}^{2} g(x) dx + \int_{2}^{6} g(x) dx + \int_{6}^{7} g(x) dx = 4 - 2\pi + \int_{6}^{7} g(x) dx,$$

we just need to determine $\int_{6}^{7} g(x) dx$. Since this is a straight line of slope 1 going up from the $x$-axis (at $x = 6$) to $y = 1$ (at $x = 7$), it describes a triangle of area

$$\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}.$$

Since this area lies above the axis, $\int_{6}^{7} g(x) dx = 1/2$, so

$$\int_{0}^{7} g(x) dx = 4 - 2\pi + \int_{6}^{7} g(x) dx = 4 - 2\pi + \frac{1}{2} = \frac{9}{2} - 2\pi \approx -1.78.$$
3. Exercise 5.2.44. Use the result of Example 3 to compute

$$
\int_{1}^{3} (2e^x - 1)dx.
$$

**Answer:** Example 3 says that $\int_{1}^{3} e^x dx = e^3 - e$, we need to use the properties of the definite integral to express the given integral in terms of $\int_{1}^{3} e^x dx$.

Now, by Property 4,

$$
\int_{1}^{3} (2e^x - 1)dx = \int_{1}^{3} 2e^x - \int_{1}^{3} 1dx.
$$

In turn, by Property 1,

$$
\int_{1}^{3} 1dx = 1(3 - 1) = 2.
$$

By Property 3,

$$
\int_{1}^{3} 2e^x dx = 2 \int_{1}^{3} e^x dx.
$$

Putting these together, then,

$$
\int_{1}^{3} (2e^x - 1)dx = 2 \int_{1}^{3} e^x dx - 2.
$$

Plugging in the value we know for $\int_{1}^{3} e^x dx$, we see that

$$
\int_{1}^{3} (2e^x - 1)dx = 2(e^3 - e) - 2 = 2(e^3 - e - 1) \approx 32.73.
$$

4. Exercise 5.3.14. Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the function

$$
h(x) = \int_{0}^{x^2} \sqrt{1 + r^3} dr.
$$

**Answer:** Make the change of variables $u = x^2$. Then

$$
h'(x) = \frac{d}{dx} \left( \int_{0}^{x^2} \sqrt{1 + r^3} dr \right) = \frac{d}{dx} \left( \int_{0}^{u} \sqrt{1 + r^3} dr \right).
$$

By the Chain Rule, this is equal to

$$
\frac{d}{du} \left( \int_{0}^{u} \sqrt{1 + r^3} dr \right) \frac{du}{dx}.
$$

Using the Fundamental Theorem and the fact that $\frac{du}{dx} = 2x$, we see that

$$
h'(x) = \sqrt{1 + u^3} (2x) = \sqrt{1 + (x^2)^3} (2x) = 2x \sqrt{1 + x^6}.
$$
5. Exercise 5.3.26. Evaluate the integral

\[ \int_{\pi}^{2\pi} \cos \theta \, d\theta. \]

**Answer:** Since \( \sin \theta \) is an antiderivative of \( \cos \theta \), the second part of the Fundamental Theorem says that

\[ \int_{\pi}^{2\pi} \cos \theta \, d\theta = \left[ \sin \theta \right]_{\pi}^{2\pi} = \sin 2\pi - \sin \pi = 0 - 0 = 0. \]

6. Exercise 5.3.36. Evaluate the integral

\[ \int_{0}^{1} 10^x \, dx. \]

**Answer:** Since

\[ \frac{d}{dx} (10^x) = 10^x \ln 10, \]

we see that

\[ \frac{10^x}{\ln 10} \]

is an antiderivative of \( 10^x \). Therefore,

\[ \int_{0}^{1} 10^x \, dx = \left[ \frac{10^x}{\ln 10} \right]_{0}^{1} = \frac{10}{\ln 10} - \frac{1}{\ln 10} = \frac{9}{\ln 10}. \]

7. Exercise 5.3.40. Evaluate the integral

\[ \int_{1}^{2} \frac{4 + u^2}{u^3} \, du. \]

**Answer:** Re-write the integral as

\[ \int_{1}^{2} \left( \frac{4}{u^3} + \frac{u^2}{u^3} \right) du = \int_{1}^{2} 4u^{-3} du + \int_{1}^{2} u^{-1} du. \]

Then, since \( u^{-2} = \frac{1}{u^2} \) is an antiderivative for \( u^{-3} \) and since \( \ln u \) is an antiderivative for \( u^{-1} \), we see that the above is equal to

\[ \left[ \frac{4}{2u^2} \right]_{1}^{2} + \left[ \ln u \right]_{1}^{2} = \left( \frac{4}{2} + 2 \right) + \left( \ln 2 - \ln 1 \right) = \frac{3}{2} + \ln 2. \]