

10. A sample of tritium-3 decayed to 94.5% of its original amount after a year.

(a) What is the half-life of tritium-3?

**Answer:** If $N(t)$ is the amount of tritium-3 relative to the original amount, we know that the general form of $N(t)$ is

$$N(t) = Ce^{kt}.$$ 

Also, we know that $N(0) = 1$, so

$$1 = N(0) = Ce^{k\cdot0} = C,$$

so $C = 1$ and we can write

$$N(t) = e^{kt}.$$ 

Also, we know $N(1) = 0.945$, so

$$0.945 = N(1) = e^{k\cdot1} = e^k.$$ 

Taking the natural log of both sides,

$$k = \ln(0.945).$$ 

Therefore,

$$N(t) = e^{\ln(0.945)t} = \left(e^{\ln(0.945)}\right)^t = (0.945)^t$$

for any $t$.

The half-life of tritium-3 is the amount of time $t_0$ such that $N(t_0) = 0.5$. Therefore, we can solve for $t_0$ from the equation

$$0.5 = N(t_0) = (0.945)^{t_0}.$$ 

Taking the natural log of both sides,

$$\ln(0.5) \ln(0.945^{t_0}) = t_0 \ln(0.945),$$

so

$$t_0 = \frac{\ln(0.5)}{\ln(0.945)} \approx 12.25.$$ 

Therefore, the half-life of tritium-3 is 12.25 years.

(b) How long would it take the sample to decay to 20% of its original amount?

**Answer:** If $t_1$ is the time it takes the sample to decay to 20% of its original amount,

$$0.2 = N(t_1) = (0.945)^{t_1},$$
meaning that (if we take the natural log of both sides),
\[ \ln(0.2) = \ln(0.945^{t_1}) = t_1 \ln(0.945), \]
so
\[ t_1 = \frac{\ln(0.2)}{\ln(0.945)} \approx 28.45 \text{ years}. \]

**Answer:**

14. A thermometer is taken from a room where the temperature is 20°C to the outdoors, where the temperature is 5°C. After one minute the thermometer reads 12°C.

(a) What will the reading on the thermometer be after one more minute?

**Answer:** From Newton’s Law of Cooling, we know that
\[ T(t) = T_s + Ce^{kt}. \]
The ambient temperature is \( T_s = 5 \)°C, whereas
\[ 20 = T(0) = 5 + Ce^{k \cdot 0} = 5 + C, \]
so \( C = 15 \). Therefore,
\[ T(t) = 5 + 15e^{kt}. \]
Moreover, we know that \( T(1) = 12 \), so
\[ 12 = T(1) = 5 + 15e^{k \cdot 1} = 5 + 15e^k, \]
so
\[ e^k = \frac{7}{15}. \]
Taking the natural log of both sides,
\[ k = \ln \left( \frac{7}{15} \right). \]
Hence,
\[ T(t) = 5 + 15e^{\ln(\frac{7}{15})t} = 5 + 15 \left( e^{\ln(\frac{7}{15})} \right)^t = 5 + \left( \frac{7}{15} \right)^t. \]
Therefore, after 2 minutes, the temperature of the thermometer will be
\[ T(2) = 5 + 15 \left( \frac{7}{15} \right)^2 = 5 + \frac{49}{15} = \frac{124}{15} = 8.266 \ldots \]

(b) When will the thermometer read 6°C?

**Answer:** The time \( t_0 \) when \( T(t_0) = 6 \) is given by
\[ 6 = T(t_0) = 5 + 15 \left( \frac{7}{15} \right)^{t_0}, \]
so
\[ \frac{1}{15} = \left( \frac{7}{15} \right)^{t_0}. \]

Taking the natural log of both sides,
\[ \ln \left( \frac{1}{15} \right) = \ln \left( \frac{7}{15} \right)^{t_0} = t_0 \ln \left( \frac{7}{15} \right). \]

Therefore,
\[ t_0 = \frac{\ln \frac{1}{15}}{\ln \frac{7}{15}} \approx 3.55, \]
so the thermometer will read 6°C after about 3 and a half minutes.

16. A freshly brewed cup of coffee has temperature 95°C in a 20°C room. When its temperature is 70°C, it is cooling at a rate of 1°C per minute. When does this occur?

**Answer:** From Newton’s Law of Cooling, the temperature of the coffee is given by
\[ T(t) = 20 + Ce^{kt}. \]

At time \( t = 0 \),
\[ 95 = T(0) = 20 + Ce^{k \cdot 0} = 20 + C, \]
meaning that \( C = 75 \) and \( T(t) = 20 + 75e^{kt} \). At some time \( t_0 \),
\[ 70 = T(t_0) = 20 + 75e^{kt_0}, \]
so
\[ 75e^{kt_0} = 50, \]
or
\[ e^{kt_0} = \frac{2}{3}. \]

Therefore,
\[ kt_0 = \ln \left( \frac{2}{3} \right). \]

We also know the rate of change of \( T \) at this time \( t_0 \):
\[ -1 = T'(t_0) = 75(e^{kt_0}k) = 75ke^{kt_0}. \]

In other words,
\[ k = \frac{-1}{75e^{kt_0}}. \]

Since \( kt_0 = \ln \left( \frac{2}{3} \right) \), we know that
\[ k = \frac{-1}{75e^{\ln \left( \frac{2}{3} \right)}} = \frac{-1}{75 \frac{2}{3}} = \frac{-1}{50}. \]

Since \( kt_0 = \ln \left( \frac{2}{3} \right) \), we know that
\[ t_0 = \frac{\ln \left( \frac{2}{3} \right)}{k} = \frac{\ln \left( \frac{2}{3} \right)}{-\frac{1}{50}} = 50 \ln \left( \frac{2}{3} \right) \approx 20.3, \]
so the cup of coffee is 70°C after just over 20 minutes.
§3.10

12. Find the differential of the functions

(a) \( y = s/(1 + 2s) \)

Answer: If \( f(s) = \frac{s}{1 + 2s} \), then, by definition,

\[
dy = f'(s)\,ds.
\]

Now,

\[
f'(s) = \frac{(1 + 2s) \cdot 1 - s \cdot 2}{(1 + 2s)^2} = \frac{1 + 2s - 2s}{(1 + 2s)^2} = \frac{1}{(1 + 2s)^2}.
\]

Therefore, the differential is

\[
dy = \frac{ds}{(1 + 2s)^2}.
\]

(b) \( y = e^{-u} \cos u \)

Answer: If \( g(u) = e^{-u} \cos u \), then, by definition,

\[
dy = g'(u)\,du.
\]

Since

\[
g'(u) = -e^{-u} \cos u + e^{-u}(-\sin u) = -e^{-u} \cos u - e^{-u} \sin u = -e^{-u}(\cos u + \sin u),
\]

we have that

\[
dy = -e^{-u}(\cos u + \sin u)\,du.
\]

18. (a) Find the differential \( dy \) of \( y = \cos x \).

Answer: By definition, if \( f(x) = \cos x \), then

\[
dy = f'(x)\,dx.
\]

Since \( f'(x) = -\sin x \), this means that

\[
dy = -\sin x\,dx.
\]

(b) Evaluate \( dy \) for \( x = \pi/3 \) and \( dx = 0.05 \).

Answer: Given the above expression for \( dy \) and knowing that \( \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \), we have that

\[
dy = -\frac{\sqrt{3}}{2}(0.05) = -\frac{\sqrt{3}}{40} \approx 0.0433.
\]

24. Use a linear approximation (or differentials) to estimate \( e^{-0.015} \).

Answer: Let \( f(x) = e^x \). If \( L(x) \) is the linearization of \( f \) at 0, then

\[
L(x) = f(0) + f'(0)(x - 0) = 1 + 1(x - 0) = 1 + x.
\]

Since \(-0.015\) is close to 0, it should be the case that

\[
e^{-0.015} \approx L(-0.015) = 1 + (-0.015) = 0.985.
\]
32. Let \( f(x) = (x - 1)^2 \), \( g(x) = e^{-2x} \), \( h(x) = 1 + \ln(1 - 2x) \).

(a) Find the linearizations of \( f \), \( g \), and \( h \) at \( a = 0 \). What do you notice? How do you explain what happened?

**Answer:** By definition, the linearization of \( f \) is

\[
 f(0) + f'(0)(x - 0) = f(0) + f'(0)x.
\]

Since \( f'(x) = 2(x - 1) \), we know that \( f(0) = 1 \) and \( f'(0) = -2 \), so the linearization of \( f \) is

\[ 1 - 2x. \]

By definition, the linearization of \( g \) is

\[
 g(0) + g'(0)(x - 0) = g(0) + g'(0)x.
\]

Since \( g'(x) = -2e^{-2x} \), we know that \( g(0) = 1 \) and \( g'(0) = -2 \), so the linearization of \( g \) is

\[ 1 - 2x. \]

By definition, the linearization of \( h \) is

\[
 h(0) + h'(0)(x - 0) = h(0) + h'(0)x.
\]

Since \( h'(x) = \frac{-2}{1 - 2x} \), we know that \( h(0) = 1 \) and \( h'(0) = -2 \), so the linearization of \( h \) is

\[ 1 - 2x. \]

We notice that all three linearizations are the same. This occurs because \( f(0) = g(0) = h(0) \) and \( f'(0) = g'(0) = h'(0) \): all three functions have the same value at 0 and their derivatives also have the same value at 0. Of course, this says nothing about the behavior of the three functions at other points.
(b) Graph $f$, $g$, and $h$ and their linear approximations. For which function is the linear approximation best? For which is it worst? Explain.

Answer:

![Graph of functions and their linear approximations](image)

Figure 1: Blue: $f$; Red: $g$; Purple: $h$; Black: linearization

From the picture, the linear approximation appears to be best for $f$ and worst for $h$.

Answer: