Math 113 HW #2 Solutions

§1.6

20. In the theory of relativity, the mass of a particle with speed $v$ is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where $m_0$ is the rest mass of the particle and $c$ is the speed of light in a vacuum. Find the inverse function of $f$ and explain its meaning.

**Answer:** We can find the inverse function by solving for $v$ in the above expression. First, multiply both sides by $\frac{\sqrt{1 - v^2/c^2}}{m}$:

$$\sqrt{1 - v^2/c^2} = \frac{m_0}{m}.$$

Squaring both sides yields

$$1 - v^2/c^2 = \frac{m_0^2}{m^2}.$$

This means that

$$v^2 = c^2 \left(1 - \frac{m_0^2}{m^2}\right),$$

so we have that

$$v = f^{-1}(m) = c \sqrt{1 - \frac{m_0^2}{m^2}}.$$

If we measure the mass of a moving particle, this expression allows us to determine the velocity of the particle.

26. Find a formula for the inverse of the function

$$y = \frac{e^x}{1 + 2e^x}.$$

**Answer:** To find the inverse, we first swap the roles of $x$ and $y$:

$$x = \frac{e^y}{1 + 2e^y}.$$

Now, the goal is to solve for $y$, so multiply both sides by $1 + 2e^y$:

$$x(1 + 2e^y) = e^y.$$

Subtracting $e^y$ from both sides yields:

$$x + 2xe^y - e^y = 0.$$

To isolate the $y$’s, first subtract $x$ from both sides:

$$e^y(2x - 1) = -x.$$
Now, dividing by $2x - 1$, we see that

$$e^y = \frac{-x}{2x - 1}.$$  

Finally, taking the natural logarithm of both sides yields

$$y = \ln \left( \frac{-x}{2x - 1} \right),$$

which is an expression for $f^{-1}(x)$.

36. Find the exact value of each expression:

(a) $e^{-2 \ln 5}$

**Answer:** First, notice that

$$-2 \ln 5 = \ln (5^{-2}) = \ln \left( \frac{1}{5^2} \right) = \ln \left( \frac{1}{25} \right).$$

Therefore, since $e^{\ln x} = x$ for any $x > 0$, we have that

$$e^{-2 \ln 5} = e^{\ln \left( \frac{1}{25} \right)} = \frac{1}{25}.$$  

(b) $\ln \left( \ln e^{10} \right)$

**Answer:** Since $\ln (e^x) = x$ for any $x$, we have that

$$\ln e^{10} = e^{10}.$$  

Therefore,

$$\ln \left( \ln e^{10} \right) = \ln (e^{10}) = 10.$$  

38. Express the quantity

$$\ln(a + b) + \ln(a - b) - 2 \ln c$$

as a single logarithm.

**Answer:** Using the properties of logarithms, we know that

$$\ln(a + b) + \ln(a - b) = \ln \left( (a + b)(a - b) \right) = \ln \left( a^2 - b^2 \right)$$

and that

$$2 \ln c = \ln \left( c^2 \right).$$

Hence,

$$\ln(a + b) + \ln(a - b) - 2 \ln c = \ln \left( a^2 - b^2 \right) - \ln \left( c^2 \right);$$

in turn, this is equal to

$$\ln \left( \frac{a^2 - b^2}{c^2} \right).$$
50. Solve each equation for $x$.

(a) $\ln(\ln x) = 1$

**Answer:** Since $e^{\ln x} = x$, we can exponentiate both sides to see that
\[ \ln x = e^1 = e. \]

Exponentiating both sides again yields
\[ x = e^e. \]

(b) $e^{ax} = C e^{bx}$, where $a \neq b$.

**Answer:** Dividing both sides by $e^{bx}$, we have that
\[ \frac{e^{ax}}{e^{bx}} = C. \]

However, the left side can be re-written, using the properties of exponentials, as $e^{ax-bx} = e^{(a-b)x}$, so we have that
\[ e^{(a-b)x} = C. \]

Now, taking the natural logarithm of both sides, we have that
\[ (a - b)x = \ln C. \]

Dividing both sides by $a - b$ gives the expression for $x$:
\[ x = \frac{\ln C}{a - b}. \]
§2.2

14. Sketch the graph of a function $f$ that satisfies all of the following conditions:

\[
\begin{align*}
\lim_{x \to 0^-} f(x) &= 1, & \lim_{x \to 0^+} f(x) &= -1, & \lim_{x \to 2^-} f(x) &= 0, \\
\lim_{x \to 2^+} f(x) &= 1, & f(2) &= 1, & f(0) &\text{ is undefined}
\end{align*}
\]

Answer:

28. Determine the infinite limit

\[
\lim_{x \to 5^-} \frac{e^x}{(x-5)^3}.
\]

Answer: Whenever $x < 5$, the expression $x - 5$ will be a negative number. Therefore, as $x \to 5^-$, we see that $x - 5$ becomes a very small negative number. Since taking the cube preserves sign and since $e^x > 0$ for all $x$, this means that

\[
\lim_{x \to 5^-} \frac{e^x}{(x-5)^3} = -\infty.
\]

40. In the theory of relativity, the mass of a particle with velocity $v$ is given by

\[
m = \frac{m_0}{\sqrt{1 - v^2/c^2}}
\]
where \( m_0 \) is the mass of the particle at rest and \( c \) is the speed of light. What happens as \( v \to c^- \)?

**Answer:** Whenever \( v < c \), the fraction \( \frac{v}{c} < 1 \). Therefore, as \( v \to c^- \), the fraction \( \frac{v^2}{c^2} \) gets closer and closer to 1, but is always less than 1. Hence, the quantity

\[
1 - \frac{v^2}{c^2}
\]

is always positive but approaches zero as \( v \to c^- \). Therefore, as \( v \to c^- \), the mass

\[
m = \frac{m_0}{\sqrt{1 - v^2/c^2}}
\]

approaches \(+\infty\).

§2.3

14. If it exists, evaluate the limit

\[
\lim_{x \to 4} \frac{x^2 - 4x}{x^2 - 3x - 4}.
\]

**Answer:** Since \( x^2 - 4x = x(x - 4) \) and since \( x^2 - 3x - 4 = (x + 1)(x - 4) \), we have that

\[
\lim_{x \to 4} \frac{x(x - 4)}{(x + 1)(x - 4)} = \lim_{x \to 4} \frac{x}{x + 1} = \frac{4}{5}.
\]

26. If it exists, evaluate the limit

\[
\lim_{t \to 0} \left( \frac{1}{t} - \frac{1}{t^2 + t} \right).
\]

**Answer:** In this case, it’s helpful to simplify the expression inside the parentheses. Since \( t^2 + t = t(t + 1) \), we have that

\[
\frac{1}{t} - \frac{1}{t^2 + t} = \frac{t + 1}{t(t + 1)} - \frac{1}{t(t + 1)} = \frac{t}{t(t + 1)}.
\]

Therefore,

\[
\lim_{t \to 0} \left( \frac{1}{t} - \frac{1}{t^2 + t} \right) = \lim_{t \to 0} \frac{t}{t(t + 1)} = \lim_{t \to 0} \frac{1}{t + 1} = 1.
\]

38. Prove that \( \lim_{x \to 0^+} \sqrt{xe^{\sin(\pi/x)}} = 0 \).

**Answer:** Since \(-1 \leq \sin(\pi/x) \leq 1\) it should be fairly easy to get upper and lower bounds for the expression \( \sqrt{xe^{\sin(\pi/x)}} \). If we can do that, then there’s a good chance we might be able to use the Squeeze Theorem.
We know that $-1 \leq \sin(\pi/x) \leq 1$ and that $e^x$ is an increasing function, so we get

$$\frac{1}{e} = e^{-1} \leq e^{\sin(\pi/x)} \leq e^1 = e.$$ 

Therefore,

$$\frac{\sqrt{x}}{e} \leq \sqrt{xe^{\sin(\pi/x)}} \leq e\sqrt{x}.$$ 

However,

$$\lim_{x \to 0^+} \frac{\sqrt{x}}{e} = \frac{1}{e} \lim_{x \to 0^+} \sqrt{x} = 0$$

and

$$\lim_{x \to 0^+} e\sqrt{x} = e \lim_{x \to 0^+} \sqrt{x} = 0.$$ 

Therefore, by the Squeeze Theorem,

$$\lim_{x \to 0^+} \sqrt{xe^{\sin(\pi/x)}} = 0.$$ 

56. If $\lim_{x \to 0} \frac{f(x)}{x^2} = 5$, find the following limits.

(a) $\lim_{x \to 0} f(x)$

Answer: The only way I can see how to do this is to re-express what we want in terms of what we know. Since we know $\lim_{x \to 0} \frac{f(x)}{x^2}$, it would be nice to express $f(x)$ in terms of $\frac{f(x)}{x^2}$. We can do this when $x \neq 0$:

$$f(x) = \frac{f(x)}{x^2} \cdot x^2.$$ 

Therefore,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left( \frac{f(x)}{x^2} \cdot x^2 \right).$$

But, using one of the limit laws, this is equal to

$$\left( \lim_{x \to 0} \frac{f(x)}{x^2} \right) \left( \lim_{x \to 0} x^2 \right) = 5 \cdot 0 = 0.$$ 

(b) $\lim_{x \to 0} \frac{f(x)}{x}$

Answer: When $x \neq 0$,

$$\frac{f(x)}{x} = \frac{f(x)}{x^2} \cdot x.$$ 

Therefore,

$$\lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \left( \frac{f(x)}{x^2} \cdot x \right).$$

Using one of the limit laws, this is equal to

$$\left( \lim_{x \to 0} \frac{f(x)}{x^2} \right) \left( \lim_{x \to 0} x \right) = 5 \cdot 0 = 0.$$