§8.1

5. Let \( a_n = \frac{2^n}{2^n+1} \). Find the values of \( a_1, a_2, a_3 \) and \( a_4 \).
   
   Answer:

   \[
   \begin{align*}
   a_1 &= 2^1 = \frac{2}{2} = 1 \\
   a_2 &= 2^2 = \frac{4}{8} = \frac{1}{2} \\
   a_3 &= 2^3 = \frac{8}{16} = \frac{1}{2} \\
   a_4 &= 2^4 = \frac{16}{32} = \frac{1}{2}
   \end{align*}
   \]

12. Let \( a_1 = 2, a_2 = -1 \) and \( a_{n+2} = \frac{a_{n+1}}{a_n} \). Find the first ten terms of the sequence.
   
   Answer:

   \[
   \begin{align*}
   a_1 &= 2 \\
   a_2 &= -1 \\
   a_3 &= \frac{a_2}{a_1} = \frac{-1}{2} \\
   a_4 &= \frac{a_3}{a_2} = \frac{-1/2}{-1} = \frac{1}{2} \\
   a_5 &= \frac{a_4}{a_3} = \frac{1/2}{1/2} = 1 \\
   a_6 &= \frac{a_5}{a_4} = \frac{1}{1} = -1 \\
   a_7 &= \frac{a_6}{a_5} = \frac{-1}{1} = -2 \\
   a_8 &= \frac{a_7}{a_6} = \frac{-2}{-1} = 2 \\
   a_9 &= \frac{a_8}{a_7} = \frac{2}{2} = -1 \\
   a_{10} &= \frac{a_9}{a_8} = \frac{-1}{2} = \frac{1}{2}
   \end{align*}
   \]
15. Find a formula for the $n$th term in the sequence $1, -4, 9, -16, 25, \ldots$.
   **Answer:** The $n$th term of the sequence is 
   $$a_n = (-1)^{n-1} \cdot n^2.$$ 

21. Find a formula for the $n$th term in the sequence $1, 0, 1, 0, 1, \ldots$.
   **Answer:** The $n$th term in the sequence is 
   $$a_n = \frac{1}{2} + \frac{(-1)^{n+1}}{2}.$$ 

38. Let 
   $$a_n = \frac{2^n - 1}{3^n}.$$ 

   Does the sequence $\{a_n\}$ converge or diverge?
   **Answer:** Note that 
   $$a_n = \frac{2^n - 1}{3^n} = \left(\frac{2}{3}\right)^n - \left(\frac{1}{3}\right)^n.$$ 

   Since $\left(\frac{2}{3}\right)^n$ and $\left(\frac{1}{3}\right)^n$ both converge to zero, $\{a_n\}$ also converges to zero.

49. Is it true that a sequence $\{a_n\}$ of positive numbers must converge if it is bounded from above? Give reasons for your answer.
   **Answer:** No, it’s not true that a bounded sequence of positive numbers must converge. For example, consider the sequence 
   $$1, 2, 1, 2, 1, 2, 1, 2, \ldots$$ 

   This sequence is bounded by 2 (that is, every term in the sequence is less than or equal to 2), but the sequence does not converge.

§8.2

2. Let 
   $$a_n = \frac{n + (-1)^n}{n}.$$ 

   Does $\{a_n\}$ converge or diverge?
   **Answer:** Note that 
   $$a_n = \frac{n + (-1)^n}{n} = 1 + \frac{(-1)^n}{n}.$$ 

   Hence, the question boils down to whether $\left\{ \frac{(-1)^n}{n} \right\}$ converges or diverges. Now, 
   $$\left| \frac{(-1)^n}{n} \right| \leq \frac{1}{n}.$$ 

   Since $\frac{1}{n} \to 0$, the Sandwich Theorem tells us that $\frac{(-1)^n}{n} \to 0$. Hence, 
   $$\frac{n + (-1)^n}{n} \to 1 + 0 = 1.$$ 

6. Let 
   $$a_n = \frac{n + 3}{n^2 + 5n + 6}.$$
Does \( \{a_n\} \) converge or diverge?

**Answer:** By L’Hôpital’s Rule,

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n + 3}{n^2 + 5n + 6} = \lim_{n \to \infty} \frac{1}{2n} = 0.
\]

14. Let

\[a_n = \left( -\frac{1}{2} \right)^n.\]

Does \( \{a_n\} \) converge or diverge?

**Answer:** Note that

\[
\left| -\frac{1}{2} \right|^n \leq \left( \frac{1}{2} \right)^n = \frac{1}{2^n}.
\]

Now, \( \frac{1}{2^n} \to 0 \), so, by the sandwich theorem, \( a_n \to 0 \).

18. Let

\[a_n = n\pi \cos(n\pi).\]

Does \( \{a_n\} \) converge or diverge?

**Answer:** Note that \( \cos(n\pi) = (-1)^n \). Hence,

\[a_n = n\pi \cos(n\pi) = (-1)^n n\pi.\]

As \( n \to \infty \), \( n\pi \to \infty \), so \( \{a_n\} \) diverges.

33. Let

\[a_n = \frac{\ln n}{n^{1/n}}.\]

Does \( \{a_n\} \) converge or diverge?

**Answer:** Note that \( n^{1/n} = \sqrt[n]{n} \to 1 \) as \( n \to \infty \) (see table page 625).

Since \( \ln n \to \infty \) as \( n \to \infty \), this means that

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{\ln n}{n^{1/n}} = \infty,
\]

so the sequence diverges.

§8.3

10. Write the first few terms of the following series, then find the sum of the series:

\[
\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}.
\]
Answer: If \( \{a_n\} \) is the sequence of terms, then

\[
\begin{align*}
    a_0 &= (-1)^0 \frac{5}{4^0} = 5 \\
    a_1 &= (-1)^1 \frac{5}{4^1} = -\frac{5}{4} \\
    a_2 &= (-1)^2 \frac{5}{4^2} = \frac{5}{16} \\
    a_3 &= (-1)^3 \frac{5}{4^3} = -\frac{5}{64} \\
    a_4 &= (-1)^4 \frac{5}{4^4} = \frac{5}{256}
\end{align*}
\]

To find the sum of the series, we re-write:

\[
\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{5}{4^{n-1}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4^{n-1} - 1}.
\]

Hence, this is a geometric series with \( a = 5 \) and \( b = \frac{-1}{4} \), so

\[
\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n} = \frac{5}{1 - (-\frac{1}{4})} = \frac{5}{\frac{3}{4}} = 4.
\]

12. Write the first few terms of the following series, then find the sum of the series:

\[
\sum_{n=0}^{\infty} \left( \frac{5}{2^n} - \frac{1}{3^n} \right).
\]

Answer: Just computing directly, we see that

\[
\begin{align*}
    \sum_{n=0}^{\infty} \left( \frac{5}{2^n} - \frac{1}{3^n} \right) &= \sum_{n=0}^{\infty} \frac{5}{2^n} - \sum_{n=0}^{\infty} \frac{1}{3^n} \\
    &= \sum_{n=1}^{\infty} \frac{5}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{3^{n-1}} \\
    &= \sum_{n=1}^{\infty} 5 \cdot \left( \frac{1}{2} \right)^{n-1} - \sum_{n=1}^{\infty} 1 \cdot \left( \frac{1}{3} \right)^{n-1} \\
    &= \frac{5}{1 - \frac{1}{2}} - \frac{1}{1 - \frac{1}{3}} \\
    &= \frac{5}{2} - \frac{1}{\frac{2}{3}} \\
    &= \frac{15}{2} - \frac{3}{2} \\
    &= \frac{17}{2}.
\end{align*}
\]
16. Use partial fractions to find the sum of the series

\[ \sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}. \]

**Answer:** First, let

\[ \frac{6}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1} \]

and solve for \( A \) and \( B \):

\[ 6 = A(2n+1) + B(2n-1). \]

Letting \( x = -\frac{1}{2} \), we see that

\[ 6 = A(0) + B(-2) = -2B, \]

so \( B = -3 \). Letting \( x = \frac{1}{2} \), we see that

\[ 6 = A(2) + B(0) = 2A, \]

so \( A = 3 \). Hence,

\[ \sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty} \left( \frac{3}{2n-1} - \frac{3}{2n+1} \right) \]

Hence, the \( n \)th partial sum is given by

\[ s_n = (3 - 1) + \left( 1 - \frac{3}{5} \right) + \left( \frac{3}{5} - \frac{3}{7} \right) + \ldots + \left( \frac{3}{2n-1} - \frac{3}{2n+1} \right) \]

\[ = 3 - \frac{3}{2n+1}. \]

Thus, as \( n \to \infty \), \( s_n \to 3 \), so

\[ \sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)} = 3. \]

23. Does the series converge or diverge?

\[ \sum_{n=0}^{\infty} \left( \frac{1}{\sqrt{2}} \right)^n. \]

**Answer:** Note that

\[ \sum_{n=0}^{\infty} \left( \frac{1}{\sqrt{2}} \right)^n = \sum_{n=1}^{\infty} \left( \frac{1}{\sqrt{2}} \right)^{n-1} = \frac{1}{1 - \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2} - 1}, \]

since \( r = \frac{1}{\sqrt{2}} < 1 \).

24. Does the series converge or diverge?

\[ \sum_{n=0}^{\infty} \left( \sqrt{2} \right)^n. \]

**Answer:** Note that

\[ \sum_{n=0}^{\infty} \left( \sqrt{2} \right)^n = \sum_{n=1}^{\infty} \left( \sqrt{2} \right)^{n-1} \]

which is a divergent geometric series, since \( r = \sqrt{2} > 1 \).
25. Does the series converge or diverge?

\[
\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{2^n}.
\]

**Answer:** Note that

\[
\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{2^n} = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{3}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{3}{2} \left( \frac{-1}{2} \right)^{n-1} = \frac{\frac{3}{2}}{1 + \frac{1}{2}} = 1
\]

since this is a geometric series with \( r = \frac{-1}{2} \), which has absolute value < 1.

46. Find the values of \( x \) for which the series

\[
\sum_{n=0}^{\infty} (-1)^n x^{-2n}
\]

converges. Also, find the sum of the series (as a function of \( x \)) for those values of \( x \).

**Answer:** Note that

\[
\sum_{n=0}^{\infty} (-1)^n x^{-2n} = \sum_{n=1}^{\infty} (-1)^{n-1} x^{-2(n-1)} = \sum_{n=1}^{\infty} x^{-2} (-x)^{n-1}.
\]

This is a geometric series with \( a = x^{-2} \) and \( r = -x \), so it converges if \( |r| = |x| < 1 \), which is to say if \( |x| < 1 \). For such values, we know, moreover, that the series converges to

\[
\frac{x^{-2}}{1 - (-x)} = \frac{\frac{1}{x^2}}{1 + x} = \frac{1}{x^2 + x^3}.
\]

77. Helga von Koch’s snowflake is a curve of infinite length that encloses a region of finite area. To see why this is so, suppose the curve is generated by starting with an equilateral triangle whose sides have length 1.

(a): Find the length \( L_n \) of the \( n \)th curve \( C_n \) and show that \( \lim_{n \to \infty} L_n = \infty \).

**Answer:** Note that \( L_1 = 1 + 1 + 1 = 3 \). Now, the length of each side of \( C_2 \) is \( \frac{1}{3} \), and there are \( 4 \cdot 3 = 12 \) of them, so

\[
L_2 = 4 \cdot 3 \cdot \frac{1}{3} = 4.
\]

The length of each side of \( C_3 \) is \( \frac{1}{3^2} \) and there are \( 4 \cdot (4 \cdot 3) = 4^2 \cdot 3 = 48 \) of them, so

\[
L_3 = 4^2 \cdot 3 \cdot \frac{1}{3^2} = \frac{4^2}{3}.
\]

In general,

\[
L_n = 4^{n-1} \cdot 3 \cdot \frac{1}{3^{n-1}} = 3 \cdot \left( \frac{4}{3} \right)^{n-1}.
\]

As \( n \to \infty \), \( \left( \frac{4}{3} \right)^{n-1} \to \infty \) since \( \frac{4}{3} > 1 \), so we see that \( L_n \to \infty \) as \( n \to \infty \).
(b): Find the area $A_n$ of the region enclosed by $C_n$ and calculate $\lim_{n \to \infty} A_n$.

**Answer:** Recall that if $T$ is an equilateral triangle with sides of length $a$, then the area of $T$ is given by $\frac{\sqrt{3}}{4}a^2$. Hence,

$$A_1 = \frac{\sqrt{3}}{4}.$$

Now, the area of $C_2$ is given by the area of $A_1$ plus the area of three equilateral triangles of side length $\frac{1}{3}$. That is,

$$A_2 = A_1 + 3 \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{3}\right)^2 = A_1 + \frac{1}{3} \cdot \frac{\sqrt{3}}{4}.$$

Now, $C_3$ has area given by the area of $C_2$ plus the area of $12 = 4 \cdot 3$ equilateral triangles of side length $\frac{1}{3^2}$. That is,

$$A_3 = A_2 + 2^2 \cdot 3 \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{3^2}\right)^2 = A_2 + \frac{1}{3} \left(\frac{4}{9}\right) \frac{\sqrt{3}}{3}.$$

In turn, to get $C_4$ we add the areas of $4 \cdot 12 = 4 \cdot (4 \cdot 3)$ equilateral triangles of side length $\frac{1}{3^3}$; i.e.

$$A_4 = A_3 + 4^2 \cdot 3 \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{3^3}\right)^2 = A_3 + \frac{1}{3} \left(\frac{4}{9}\right) \left(\frac{4}{3}\right) \frac{\sqrt{3}4}{3}.$$

Then

$$A_5 = A_4 + 4^3 \cdot 3 \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{3^4}\right) = A_4 + \frac{1}{3} \left(\frac{4}{9}\right) \left(\frac{4}{3^2}\right) \frac{\sqrt{3}4}{3^4}.$$

Hence, the area of the infinite snowflake is given by

$$\sum_{n=1}^{\infty} A_n = \frac{\sqrt{3}}{4} + \sum_{n=2}^{\infty} \frac{\sqrt{3}}{12} \left(\frac{4}{9}\right)^{n-2}$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \sum_{n=1}^{\infty} \left(\frac{4}{9}\right)^{n-1}$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \frac{\frac{4}{9}}{1 - \frac{4}{9}}$$

$$= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{20}$$

$$= \frac{2\sqrt{3}}{5}$$