3. Let $M$ be a complete Riemannian manifold with non-positive sectional curvature. Prove that
$$|(d\exp_p)_v(w)| \geq |w|,$$
for all $p \in M$, all $v \in T_pM$ and all $w \in T_v(T_pM)$.

**Proof.** Let $\gamma$ be a geodesic on $M$ with $\gamma(0) = p$, $\gamma'(0) = v$. Define the Jacobi field $J(t) = (d\exp_p)_tv(w)$. Then $J(0) = 0$ and
$$J'(0) = \left. \frac{D}{dt} (t(d\exp_p)_tv(w)) \right|_{t=0} = (d\exp_p)_tv(w) + 0 \cdot \left. \left( \frac{d}{dt} (d\exp_p)_tv(w) \right) \right|_{t=0} = w.$$

Hence, $\langle J'(0), \gamma'(0) \rangle = \langle w, v \rangle$. Now, let $\tilde{M} = T_pM$, which is linearly isometric to $\mathbb{R}^n$ with the usual inner product, so the sectional curvatures are all equal to all sectional curvatures of $M$. Let $\tilde{\gamma}(t) = tv$; then $\tilde{\gamma}$ is a geodesic in $\tilde{M}$ and, if we define $\tilde{J}(t) = tw$, then $\tilde{J}$ is a Jacobi field. Moreover, $\langle \tilde{J}'(0), \tilde{\gamma}'(0) \rangle = \langle w, v \rangle$. Therefore, by the Rauch Comparison Theorem, $|J(t)| \geq |	ilde{J}(t)|$ for all $t$. In particular, for $t = 1$,
$$|(d\exp_p)_v(w)| = |J(1)| \geq |	ilde{J}(1)| = |w|.$$ 

□

4. 

(a): Let $C \subset \mathbb{R}^2$ be a regular curve. Show that the focal set $F(C) \subset \mathbb{R}^2$ of $C$ is obtained by taking, on the positive normal $n$ at $p \in C$ a length equal to $1/k$, where $k$ is the curvature of $C$ at $p$.

**Proof.** Let $x(s)$ parametrize $C$. Recall that $\exp^\perp(t, s) = x(t) + tn(s)$. If the tangent $T$ and the normal $n$ form a basis for $\mathbb{R}^2$, then
$$d \exp^\perp = \begin{pmatrix}
\langle \frac{\partial x}{\partial s} + t\frac{\partial n}{\partial s}, \frac{\partial x}{\partial s} \rangle & \langle n, \frac{\partial x}{\partial s} \rangle \\
\langle \frac{\partial x}{\partial s} + t\frac{\partial n}{\partial s}, n \rangle & \langle n, n \rangle 
\end{pmatrix} = \begin{pmatrix}
\langle \frac{\partial x}{\partial s} + t\frac{\partial n}{\partial s}, \frac{\partial x}{\partial s} \rangle & 0 \\
\langle \frac{\partial x}{\partial s} + t\frac{\partial n}{\partial s}, n \rangle & \langle n, n \rangle 
\end{pmatrix}.$$

Since $\langle n, n \rangle \neq 0$, $d \exp^\perp$ is singular (and $C$ has a focal point) if and only if
$$\langle \frac{\partial x}{\partial s} + t\frac{\partial n}{\partial s}, \frac{\partial x}{\partial s} \rangle = 0.$$
By the Frenet equations, \( \frac{\partial n}{\partial s} = -\kappa(s)T(s) \), so

\[
\frac{\partial x}{\partial s} + t \frac{\partial n}{\partial s} = T(s) - t\kappa(s)T(s) = (1 - t\kappa(s))T(s)
\]

which is zero precisely when \( 1 - t\kappa(s) = 0 \): i.e., when \( t = \frac{1}{\kappa(s)} \), exactly as predicted. Note that \( \left| \frac{\partial x}{\partial s} \right| = 0 \) precisely when \( \kappa(s) = 0 \), so we see that the above entirely characterizes the focal points of \( C \).

\( \Box \)

(b): Show that the focal set of the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) is given by

\[
\left\{ (x, y) \in \mathbb{R}^2; (ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3} \right\}
\]

Proof. Parametrize the ellipse by \( \alpha(s) = (a \cos s, b \sin s) \) (note that, in spite of the notation, this curve is not parametrized by arc length). Then \( \alpha'(s) = (-a \sin s, b \cos s) \). If we let \( n(s) = (-b \cos s, -a \sin s) \), then \( n \) is normal to the ellipse. Now, \( n'(s) = (b \sin s, -a \cos s) \), so

\[
d \exp^\perp = \begin{pmatrix} a^2 \sin^2 s + b^2 \cos^2 s \quad 0 \\ (b^2 - a^2) t \sin s \cos s \\ b^2 \cos^2 s + a^2 \sin^2 s \end{pmatrix},
\]

which has a critical point when \( t = \frac{a^2 \sin^2 s + b^2 \cos^2 s}{ab} \). Therefore, the focal locus of the ellipse is given by

\[
\alpha(s) + \frac{a^2 \sin^2 s + b^2 \cos^2 s}{ab} n(s) = \begin{pmatrix} a \cos s - \frac{a^2 \sin^2 s + b^2 \cos^2 s}{ab} b \cos u, \\ b \sin s - \frac{a^2 \sin^2 s + b^2 \cos^2 s}{ab} a \sin s \end{pmatrix} = \begin{pmatrix} \frac{a^2 - a^2 \sin^2 s - b^2 \cos^2 s}{a} \cos s, \frac{b^2 - a^2 \sin^2 s - b^2 \cos^2 s}{b} \sin s \\ \frac{b^2 - a^2 \cos^3 s}{a}, \frac{a^2 - b^2}{b} \sin^3 s \end{pmatrix}.
\]

Since

\[
\left( (b^2 - a^2) \cos^3 s \right)^{2/3} + \left( (a^2 - b^2) \sin^3 s \right)^{2/3} = (a^2 - b^2)^{2/3},
\]

we see that the focal locus of the ellipse is given by

\[
\left\{ (x, y) \in \mathbb{R}^2; (ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3} \right\}.
\]

\( \Box \)

(c): Show that the focal set of the curve

\[ t \mapsto (\cos t + t \sin t, -\sin t + t \cos t) \]

is the circle \( t \mapsto (\cos t, -\sin t) \).
Proof. Let \( \alpha(s) = (\cos s + s \sin s, -\sin s + s \cos s) \) (again, despite notation, not necessarily parametrized by arc length). Then \( \alpha'(s) = (s \cos s, -s \sin s) \). If we let \( n(s) = (\sin s, \cos s) \), then \( n \) is normal to the spiral. Now,
\[
\alpha'(s) = (\cos s, -\sin s),
\]
so
\[
d \exp^\perp = \begin{pmatrix} s^2 + ts & 0 \\ 0 & 1 \end{pmatrix},
\]
which has critical points when \( t = -s \). Therefore, the focal locus of the spiral is given by
\[
\alpha(s) - sn(s) = (\cos s + s \sin s - s \sin s, -\sin s + s \cos s - s \cos s) = (\cos s, -\sin s),
\]
as expected. \( \square \)

6.

What follows is a slight generalization of Sturm’s Comparison Theorem. We present the theorem in geometric form.

Let \( M^2 \) be a complete Riemannian manifold of dimension 2, and let \( \gamma : [0, \infty) \to M^2 \) be a geodesic. Let \( J(t) \) be a Jacobi field along \( \gamma \) with \( J(0) = J(t_0) = 0, t_0 \in (0, \infty) \), and \( J(t) \neq 0, t \in (0, t_0) \). Then \( J \) is a field normal to \( \gamma \) and can be written \( J(t) = f(t)e_2(t) \), where \( e_2(t) \) is the parallel transport of a unit vector \( e_2 \in T_{\gamma(0)}M \) with \( e_2 \perp \gamma'(0) \). Because \( J \) is a Jacobi field,
\[
f''(t) + K(t)f(t) = 0,
\]
where \( K \) is the Gaussian curvature of \( M^2 \). Assume that
\[
K(t) \leq L(t),
\]
where \( L \) is a differentiable function on \( [0, \infty) \). Prove that any solution of the equation
\[
\tilde{f}''(t) + L(t)\tilde{f}(t) = 0
\]
has a zero on \( [0, t_0] \), that is, there exists \( t_1 \in [0, t_0] \) with \( \tilde{f}(t_1) = 0 \).

Proof. Suppose \( \tilde{f}(t) \) is a solution to the given equation and that \( \tilde{f}(t) \neq 0 \) for all \( t \in [0, t_0] \). Since \( f'' + Kf = 0 \) and \( \tilde{f}'' + L\tilde{f} = 0 \),
\[
0 = \int_0^{t_0} \left[ \tilde{f}(f'' + Kf) - f(\tilde{f}'' + Lf) \right] dt
= \int_0^{t_0} (\tilde{f} f'' - f \tilde{f}'') dt + \int_0^{t_0} (K - L)f \tilde{f} dt
= [\tilde{f} f' - f \tilde{f}']_0^{t_0} - \int_0^{t_0} (\tilde{f} f' - f \tilde{f}') dt + \int_0^{t_0} (K - L)f \tilde{f} dt
= \tilde{f}(t_0)f'(t_0) - \tilde{f}(0)f'(0) + \int_0^{t_0} (K - L)f \tilde{f} dt.
\]

(1)
Now, since $\tilde{f}(t) \neq 0$ for all $t \in [0, t_0]$, then either $\tilde{f} > 0$ on this interval or $\tilde{f} < 0$ on this interval. Similarly, since $t_0$ is the first conjugate point of $\gamma(0)$ along $\gamma$, $f > 0$ on $(0, t_0)$ or $f < 0$ on $(0, t_0)$. If $\tilde{f} > 0$ and $f > 0$ on $(0, t_0)$, then, since $f(0) = 0 = f(t_0)$, $f'(0) > 0$ and $f'(t_0) < 0$. However, since $K \leq L$,

$$\int_0^{t_0} (K - L) \tilde{f} dt + \tilde{f}(t_0)f'(t_0) - \tilde{f}(0)f'(0) \leq \tilde{f}(t_0)f'(t_0) - \tilde{f}(0)f'(0) < 0,$$

contradicting (1) above. The other cases ($\tilde{f} > 0$ and $f < 0$, etc.) follow similarly.

7.

Let $M^2$ be a complete Riemannian manifold of dimension two and let $\gamma : [0, \infty) \to M^2$ be a geodesic with $\gamma(0) = p$. Let $K(s)$ be the Gaussian curvature of $M^2$ along $\gamma$. Assume that:

(2) $$\int_0^\infty K(s) ds \leq \frac{1}{4(t+1)}, \quad \text{for all } t \geq 0,$$

in the sense that the integral converges and has the bound indicated.

(a): Define

$$\omega(t) = \int_t^\infty K(s) ds + \frac{1}{4(t+1)},$$

and show that $\omega'(t) + (\omega(t))^2 \leq -K(t)$.

Proof. First, note that

$$\omega(t) = \int_0^\infty K(s) ds - \int_0^t K(s) ds + \frac{1}{4(t+1)},$$

so, by the Fundamental Theorem of Calculus,

$$\omega'(t) = -K(t) - \frac{1}{4(t+1)^2}.$$

Now,

$$(\omega(t))^2 = \left(\int_t^\infty K(s) ds\right)^2 + \frac{1}{2(t+1)} \int_t^\infty K(s) ds + \frac{1}{16(t+1)^2},$$

so

$$\omega'(t) + (\omega(t))^2 = -K(t) - \frac{1}{4(t+1)^2} + \left(\int_t^\infty K(s) ds\right)^2 + \frac{1}{2(t+1)} \int_t^\infty K(s) ds + \frac{1}{16(t+1)^2}$$

$$= -K(t) - \frac{3}{16(t+1)^2} + \left(\int_t^\infty K(s) ds\right)^2 + \frac{1}{2(t+1)} \int_t^\infty K(s) ds$$

$$\leq -K(t) - \frac{3}{16(t+1)^2} + \left(\frac{1}{4(t+1)}\right)^2 + \frac{1}{2(t+1)} \cdot \frac{1}{4(t+1)}$$

$$= -K(t).$$
(b): For $t \geq 0$, put $\omega'(t) + (\omega(t))^2 = -L(t)$ (hence $L(t) \geq K(t)$) and define
\[ \tilde{f}(t) = \exp \left( \int_0^t \omega(s)ds \right), \quad t \geq 0. \]
Show that
\[ \tilde{f}''(t) + L(t)\tilde{f}(t) = 0, \quad \tilde{f}(0) = 1. \]

**Proof.** By definition,
\[ \tilde{f}'(t) = \frac{d}{dt} \left( \int_0^t \omega(s)ds \right) \exp \left( \int_0^t \omega(s)ds \right) = \omega(t)\tilde{f}(t), \]
so
\[ \tilde{f}''(t) = \omega'(t)\tilde{f}(t) + (\omega(t))^2\tilde{f}(t). \]
Therefore,
\[ \tilde{f}''(t) + L(t)\tilde{f}(t) = \omega'(t)\tilde{f}(t) + (\omega(t))^2\tilde{f}(t) + (-\omega'(t) + (\omega(t))^2)\tilde{f}(t) = 0. \]
Also, $\tilde{f}(0) = \exp(0) = 1$, as desired. \qed

(c): Observe that $\tilde{f}(t) > 0$ and use the oscillation theorem of Sturm to show that there does not exist a Jacobi field $J(s)$ on $\gamma(s)$ with $J(0) = 0$ and $J(s_0) = 0$, for some $s_0 \in (0, \infty)$. Therefore, the condition (2) implies that there do not exist conjugate points to $p$ along $\gamma$.

**Proof.** Since the exponential is always positive, $\tilde{f}(t) > 0$. Now, suppose there exists a Jacobi field $J(s)$ on $\gamma(s)$ such that $J(0) = 0$ and $J(s_0) = 0$ for some $s_0 \in (0, \infty)$. Then, by problem 6 above, since $\tilde{f}$ is a solution of the equation $\tilde{f}''(s) + L(s)\tilde{f}(s) = 0$ and $K(s) \leq L(s)$, $\tilde{f}$ must have a zero on $[0, s_0]$. However, we just said that $\tilde{f}$ is strictly positive, so this is impossible. Therefore, we conclude that there must not exist such a Jacobi field, meaning that there are no conjugate points to $p$ along $\gamma$. \qed

(B)

Let $S^n(1)$ denote the $n$-sphere of radius 1 in $\mathbb{R}^{n+1}$. Consider the product metric on $S^3(1) \times S^1(1)$.

(a): What can one say about the curvature of this metric?

**Answer:** Let $M = S^3(1) \times S^1(1)$ and, for any $p \in M$ and $V \in T_pM$, let us denote by $V_3$ the component of $V$ parallel to $S^3$ and by $V_1$ the component of $V$ parallel to $S^1$. Now, for orthonormal $X, Y \in T_pM$,
\[ K(X, Y) = \langle R(X, Y)X, Y \rangle = \langle R(X_1 + X_3, Y_1 + Y_3)(X_1 + X_3), Y_1 + Y_3 \rangle. \]
Since everything in this expression is linear, we can completely split up all the terms. Moreover, since $S^1$ is only one-dimensional, $X_1$ and $Y_1$ must be parallel, so $R(X_1,Y_1) \equiv 0$. Also, if we think of $M$ as a bundle over $S^3(1)$, $X_1$ is vertical and $Y_3$ is horizontal, so $R(X_1,Y_3) \equiv 0$ and similarly for $R(Y_1,X_3)$, $R(X_1,X_3)$, $R(Y_1,Y_3)$, etc. Therefore, by rearranging according to the usual formulas, all of the terms in the above expression vanish except for $\langle R(X_3,Y_3)X_3,Y_3 \rangle$.

Hence,
\[
K(X,Y) = \langle R(X_3,Y_3)X_3,Y_3 \rangle = |X_3 \wedge Y_3|^2,
\]
which varies from 0 to 1. In essence, we’re taking the unit square defined by $X$ and $Y$, projecting it onto $S^3(1)$ and whatever the area of the projection is the sectional curvature of the plane spanned by $X$ and $Y$.

As for scalar curvature, let $z_1, z_2, z_3, z_4$ be an orthonormal basis for $T_pM$, where $z_1, z_2, z_3 \in T_pS^3$ and $z_4 \in T_pS^1$. Then
\[
K(p) = \frac{1}{12} \sum_{ij} \langle R(z_i,z_j)z_i,z_j \rangle;
\]
the summands are zero whenever $z_4$ is involved and 1 otherwise. There are 6 terms involving $z_4$, so $K(p) = \frac{6}{12} = \frac{1}{2}$.

For Ricci curvature, the Ricci curvature in an arbitrary direction is 0 in the $S^1$ component and more in the $S^3$ directions. Without loss of generality, assume $v \in T_pM$ is given by $v = (\sqrt{1-d^2}, 0, 0, d)$ where the first three terms are the $S^3$ directions and the fourth is the $S^1$ direction. Complete this to an orthonormal basis: $w^1 = (0,1,0,0)$, $w^2 = (0,0,1,0)$, $w^3 = (-d,0,0,\sqrt{1-d^2})$. Then
\[
\text{Ric}_p(v) = \frac{1}{3} \left( \langle R(v, w^1)v, w^1 \rangle + \langle R(v, w^2)v, w^2 \rangle + \langle R(v, w^3)v, w^3 \rangle \right)
= \frac{1}{3} \left( \langle w^1, w^1 \rangle \langle v, v \rangle - \langle v, w^1 \rangle \langle w^1, v \rangle + \langle w^2, w^2 \rangle \langle v, v \rangle - \langle v, w^2 \rangle \langle w^2, v \rangle + \langle w^3, w^3 \rangle \langle v, v \rangle - \langle v, w^3 \rangle \langle w^3, v \rangle \right)
= \frac{1}{3} \left( (1-d^2) - 0 + (1-d^2) - 0 + d^2(1-d^2) - d^2(1-d^2) \right)
= \frac{2}{3} (1-d^2)
\]
since only the parts in the $S^3$ direction are relevant and $S^3$ has constant sectional curvature.

\[\bullet\]

(b): What are the geodesics in this metric?

\textbf{Answer:} As in any product manifold, geodesics in $M$ are of the form $(\gamma_1, \gamma_3)$, where $\gamma_1$ is a geodesic in $S^1(1)$ and $\gamma_3$ is a geodesic in $S^3(1)$. 

(c): What is the first conjugate locus and cut locus of any point?

**Answer:** In the following schematic, the boundary of the picture forms the cut locus of the center point $p$, so $C_m(p) = S^3 \vee S^1$.

To determine the conjugate locus, we know that the conjugate locus of $S^3(1) \times S^1(1)$ is the image of the conjugate locus of the universal cover, $S^3(1) \times \mathbb{R}$. As we see in the following schematic picture, this means the conjugate locus of $p$ is $S^1$.

(C)

Visualize $\mathbb{R}P^2$ inside $\mathbb{C}P^2$.

**Answer:** Thinking of $\mathbb{C}P^2$ as complex lines in $\mathbb{C}^3 \simeq \mathbb{R}^6$, we know each complex line meets the unit sphere $S^5$ in a great circle, which gives an induced Hopf map $h : S^5 \to \mathbb{C}P^2$. Now, think of $z_i = x_i + iy_i$; then $x_1, x_2, x_3, y_1, y_2, y_3$ serve as coordinates of $\mathbb{R}^6$; if we consider the real 3-plane determined by $x_1, x_2, x_3$, then this 3-plane intersects the unit 5-sphere in
a great 2-sphere $S^2$. Under the map $h$, antipodal points on this $S^2$ are identified, so $h(S^2) = \mathbb{RP}^2 \subset \mathbb{CP}^2$.

The isometries of $\mathbb{CP}^2$ certainly contain $U(3)/e^{i\theta} = SU(3)$, since $U(3)$ takes complex lines to complex lines and the circle action takes each complex line to itself. Moreover, “complex conjugation”, that is, the map $(z_1, z_2, z_3) \mapsto (\bar{z}_1, \bar{z}_2, \bar{z}_3)$ takes complex lines to complex lines, so we guess that the group of isometries of $\mathbb{CP}^2$ is $SU(3) \cup$ (complex conjugation). Note that this group of isometries is transitive and isotropic.

By the transitivity of the isometries of $\mathbb{CP}^2$, we can look at just a single point on $\mathbb{RP}^2$ to investigate further; let $p = (1 : 0 : 0)$. Recall from last week that $K(X, Y) = 1 + 3 \cos^2 \phi$ where $\cos \phi = \langle X, iY \rangle$, where $X$ and $Y$ are horizontal lifts to $TS^5$ of $X, Y \in T\mathbb{CP}^2$. Specifying to the case of $\mathbb{RP}^2$, if $q \in h|^{-1}_S(p)$ (i.e. $q = (1, 0, 0)$ or $(-1, 0, 0)$), then we can just think of $T_qS^2$ as the 2-plane perpendicular to $q$ (thought of as a vector in the 3-plane described above). Letting $q = (1, 0, 0)$, then $T_qS^2$ is spanned by $(0, 1, 0)$ and $(0, 0, 1)$. Hence,

$$\cos \phi = \langle (0, 1, 0), i(0, 0, 1) \rangle = \langle (0, 1, 0), (0, 0, i) \rangle = 0,$$

so the sectional curvature at $p$ is $1 + 3 \cos^2 \phi = 1$.

Now, suppose $p \in \mathbb{CP}^2$ and $X, Y \in T_p\mathbb{CP}^2$ be orthonormal such that $K(X, Y) = 1$. Then, since the isometries of $\mathbb{CP}^2$ are transitive, there exists an isometry $f$ of $\mathbb{CP}^2$ such that $f((1 : 0 : 0)) = p$. Since the isometries are isotropic, we may as well assume $f_s \circ h_s(0, 1, 0) = X$. Thinking of the horizontal part of $T_qS^5$ as the 4-plane perpendicular to the complex line associated with $h(q)$, then, since $\langle X, iY \rangle = 0$, $(f^{-1})_sY = \pm(0, 0, 1)$ or $\pm(0, 0, i)$. In the first case, the 2-plane spanned by $X$ and $Y$ is tangent to $f(\mathbb{RP}^2)$; in the second case, it’s not. On the other hand, in the second case, the 2-plane spanned by $X$ and $Y$ is tangent to $f(\mathbb{RP}^2)$ where $\mathbb{RP}^2$ is the $\mathbb{RP}^2$ given (as above) by intersecting the 3-plane spanned by $x_1, x_2, y_3$ with $S^5$, so it is tangent to an $\mathbb{RP}^2$.

**Extra Problem**

A point in $\mathbb{CP}^2$ is chosen at random, and then a tangent 2-plane at the point is chosen at random. We know the sectional curvature can be any number between 1 and 4. Suppose we are told that it is either 1 or 4. Which of these two possibilities is more likely, and why?

**Answer:** Let $p \in \mathbb{CP}^2$. Then $p$ corresponds to a complex line in $\mathbb{C}^3$. By a suitable rotation of $\mathbb{C}^3$, we may as well assume $p$ corresponds to the complex line given by the first factor in $\mathbb{C}^3 = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$. This plane corresponds to $(1 : 0 : 0) \in \mathbb{CP}^2$ in homogeneous coordinates, so we may as well assume $p = (1 : 0 : 0)$. Now, I like to visualize $T_p\mathbb{CP}^2$ as the set of all vectors in $\mathbb{C}^3$ perpendicular to the complex line corresponding to the point $p$. Hence, if $u, v \in T_p\mathbb{CP}^2$, then we can think of $u, v$ as vectors in $0 \times \mathbb{C} \times \mathbb{C} \simeq \mathbb{C}^2$. 
Now, we know from problem 12 on HW #4 that $K(u, v) = 1 + \cos^2 \phi$ where $\cos \phi = \langle \bar{u}, i \bar{v} \rangle$ and $\bar{u}$ and $\bar{v}$ are the horizontal lifts of an orthonormal pair $u$ and $v$, respectively, to $TS^5$. For $q \in S^5$, we can think of $T_qS^5$ as the 5-plane perpendicular to $q$ thought of as a vector in $\mathbb{R}^6 \cong \mathbb{C}^3$. Now, since $p = (1 : 0 : 0)$, we know that $q = (1, 0, 0)$ lies in the fiber over $p$ and, in the above interpretation,

$$T_qS^5 = \{(i y_1, x_2 + i y_2, x_3 + i y_3) \in \mathbb{C}^3 \}.$$

The vertical part of $T_qS^5$ is just $\{(i y_1, 0, 0) \}$ and the horizontal part is $\{(0, x_2 + i y_2, x_3 + i y_3) \}$. Hence, for $u, v \in T_pCP^2$ thought of as $(u_1, u_2), (v_1, v_2) \in \mathbb{C}^2$, $\bar{u} = (0, u_1, u_2)$ and $\bar{v} = (0, v_1, v_2)$.

Now, suppose $u, v \in T_pCP^2$ are an orthonormal pair such that $K(u, v) = 1$ or 4, so either $\langle \bar{u}, i \bar{v} \rangle = \cos \phi = 0$ or $\langle \bar{u}, i \bar{v} \rangle = \cos \phi = 1$. In other words, either $u$ is parallel to $iv$ in the above visualization, or $u$ is perpendicular to $iv$. Since the space of vectors parallel to $iv$ is 1-dimensional while the space of vectors perpendicular to $iv$ is 3-dimensional, we expect that the latter should be much more likely and so $K(u, v) = 1$ should be more likely than $K(u, v) = 4$.

To see this rigorously, suppose $u = (a_1 + i b_1, a_2 + i b_2)$. If $v = \pm i u$, then

$$\cos \phi = \langle \bar{u}, i \bar{v} \rangle = \langle \bar{u}, \mp \bar{u} \rangle$$

$$= \mp 1.$$

On the other hand, supposing $b_2 \neq 0$ (if it is, perform the below calculation with $v = \pm (0, i)$), then if we let $v = \pm \frac{1}{1 + \frac{b_2}{b_1}} \left(1, -\frac{b_1}{b_2}\right)$, we see that

$$\cos \phi = \langle \bar{u}, i \bar{v} \rangle = \langle (a_1, b_1, a_2, b_2), \pm \frac{1}{1 + \frac{b_2}{b_1}} \left(0, 1, 0, -\frac{b_1}{b_2}\right) \rangle$$

$$= \pm \frac{1}{1 + \frac{b_2}{b_1}} (b_1 - b_1)$$

$$= 0.$$

Also, supposing $a_2 \neq 0$ (if it is, perform the below calculation with $v = \pm (0, 1)$), then if we let $v = \pm \frac{1}{1 + \frac{a_1}{a_2}} \left(i, -i \frac{a_1}{a_2}\right)$, we see that

$$\cos \phi = \langle \bar{u}, i \bar{v} \rangle = \langle (a_1, b_1, a_2, b_2), \pm \frac{1}{1 + \frac{a_2}{a_1}} \left(-1, 0, \frac{a_1}{a_2}, 0\right) \rangle$$

$$= \pm \frac{1}{1 + \frac{a_2}{a_1}} (-a_1 + a_1)$$

$$= 0.$$
The above three possibilities for $v$ are pairwise orthonormal, so any vector orthonormal to $u$ in $T_p\mathbb{CP}^2$ is a linear combination of them. Hence, we see that the only $v \in T_p\mathbb{CP}^2$ for which $K(u, v) = 4$ are $v = \pm iu$, whereas $K(u, v) = 1$ for any norm 1 linear combination of $\frac{1}{\sqrt{1+b_1^2}} (1, -\frac{b_1}{b_2})$ and $\frac{1}{\sqrt{1+a_1^2}} (i, -i\frac{a_1}{a_2})$.

Thus, we conclude that, if we choose a tangent 2-plane to $p$, then it is much more likely that the sectional curvature at that 2-plane is 1 than 4.