3.

Let $S$ be the “saddle surface” $z = y^2 - x^2$, officially known as a hyperbolic paraboloid. Parametrize $S$ by the map $X : \mathbb{R}^2 \to S \subset \mathbb{R}^3$, $X(u, v) = (u, v, v^2 - u^2)$, which is the usual way to parametrize the graph of a function from $\mathbb{R}^2 \to \mathbb{R}^3$. Choose $N(u, v)$ to be the roughly upward pointing unit normal vector to $S$ at the point $X(u, v)$.

(1) Show that:

Proof. Since $X(u, v) = (u, v, v^2 - u^2)$, $X_u = (1, 0, -2u)$ and $X_v = (0, 1, 2v)$. Then

$$N(u, v) = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{(2u, -2v, 1)}{|(2u, -2v, 1)|} = \frac{1}{\sqrt{4u^2 + 4v^2 + 1}}(2u, -2v, 1).$$

Then

$$dN_p(X_u) = N_u = \frac{1}{4u^2 + 4v^2 + 1} \left( 2\sqrt{4u^2 + 4v^2 + 1} + \frac{8u^2}{\sqrt{4u^2 + 4v^2 + 1}}, \frac{8uv}{\sqrt{4u^2 + 4v^2 + 1}}, \frac{-4u}{\sqrt{4u^2 + 4v^2 + 1}} \right)$$

$$= \frac{1}{(4u^2 + 4v^2 + 1)^{3/2}}(8v^2 + 2, 8uv, -4u).$$

Also,

$$dN_p(X_v) = N_v = \frac{1}{4u^2 + 4v^2 + 1} \left( \frac{-8uv}{\sqrt{4u^2 + 4v^2 + 1}}, -2\sqrt{4u^2 + 4v^2 + 1} + \frac{8v^2}{\sqrt{4u^2 + 4v^2 + 1}}, \frac{-4v}{\sqrt{4u^2 + 4v^2 + 1}} \right)$$

$$= \frac{1}{(4u^2 + 4v^2 + 1)^{3/2}}(-8uv, -8u^2 - 2, -4v).$$

□

(2) Check that $N_u$ and $N_v$ are both orthogonal to $N$. 

Proof. Note that
\[ \langle N, N_u \rangle = \frac{1}{(4u^2 + 4v^2 + 1)^2} (2u(8v^2 + 2) - 2v(8uv) - 4u) = 0 \]
and
\[ \langle N, N_v \rangle = \frac{1}{(4u^2 + 4v^2 + 1)^2} (2u(-8uv) - 2v(-8u^2 - 2) - 4v) = 0, \]
so \( N_u \) and \( N_v \) are orthogonal to \( N \). □

(3) Show that at \( p = \text{origin} \), the map \( dN_p : T_pS \to T_pS \) is given by the matrix
\[
\begin{pmatrix}
2 & 0 \\
0 & -2
\end{pmatrix}
\]
with respect to the basis \( X_u = (1, 0, 0), X_v = (0, 1, 0) \) of \( T_{\text{origin}}S \).

Proof. At the origin, \( T_pS \) lies in the \( uv \)-plane, so, with the given basis, we can ignore the \( z \)-coordinate of \( dN_p \). Hence,
\[
dN_p = \begin{pmatrix}
8v^2 + 2 & 8uv \\
-8uv & -8u^2 - 2
\end{pmatrix} = \begin{pmatrix}
2 & 0 \\
0 & -2
\end{pmatrix},
\]
since \( u = v = 0 \) at the origin. □

6.

Verify the calculations for \( X_u, X_v, N, N_u, N_v \), etc. for the helicoid
\[ X(u, v) = (u \cos v, u \sin v, v). \]

Proof. First, note that
\[ X_u = (\cos v, \sin v, 0) \] and \[ X_v = (-u \sin v, u \cos v, 1). \]

Hence,
\[
N(u, v) = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{(\sin v, - \cos v, u \cos^2 v + u \sin^2 v)}{|(\sin v, - \cos v, u \cos^2 v + u \sin^2 v)|} = \frac{1}{\sqrt{1 + u^2}}(\sin v, - \cos v, u).
\]

Therefore,
\[
N_u = \frac{1}{1 + u^2} \left( \frac{-u \sin v}{\sqrt{1 + u^2}}, \frac{u \cos v}{\sqrt{1 + u^2}}, \sqrt{1 + u^2} - \frac{u^2}{\sqrt{1 + u^2}} \right)
\]
\[
= \frac{1}{(1 + u^2)^{3/2}}(-u \sin v, u \cos v, 1)
\]
\[
= \frac{X_v}{(1 + u^2)^{3/2}}
\]
and

\[ N_v = \frac{1}{\sqrt{1 + u^2}}(\cos v, \sin v, 0) = \frac{X_u}{\sqrt{1 + u^2}}. \]

Thus,

\[ \langle N_u, X_u \rangle = \frac{1}{(1 + u^2)^{3/2}}(-u \sin v \cos v + u \sin v \cos v) = 0, \]

\[ \langle N_u, X_v \rangle = \frac{1}{(1 + u^2)^{3/2}}(u^2 \sin^2 v + u^2 \cos^2 v + 1) = \frac{1 + u^2}{(1 + u^2)^{3/2}} = \frac{1}{\sqrt{1 + u^2}}, \]

\[ \langle N_v, X_u \rangle = \frac{1}{\sqrt{1 + u^2}}(\cos^2 v + \sin^2 v) = \frac{1}{\sqrt{1 + u^2}}, \]

and

\[ \langle N_v, X_v \rangle = \frac{1}{\sqrt{1 + u^2}}(-u \sin v \cos v + u \sin v \cos v) = 0. \]

\[ \square \]

7.

On a sphere of any radius, consider a great circle and a smaller circle which are tangent to one another at some point. Let \( \kappa_G \) be the curvature of the great circle and \( \kappa_S \) be the curvature of the small circle. Let \( \theta \) denote the angle between their principal normals at the common point. Show that

\[ \kappa_S = \kappa_G / \cos \theta. \]

**Proof.** Let \( r \) be the radius of a circle. Then \( \alpha(\theta) = (r \cos \theta, r \sin \theta) \) is an arc-length parametrization of the circle. Hence,

\[ \alpha'' = \left( -\frac{1}{r^2} \cos \theta, -\frac{1}{r^2} \sin \theta \right), \]

so

\[ \kappa = |\alpha''| = \sqrt{\frac{1}{r^2}} = \frac{1}{r}. \]

Now, suppose \( R \) is the radius of the sphere. Then \( R \) is also the radius of \( G \), meaning that \( \kappa_G = \frac{1}{R} \). Now, suppose \( r \) is the radius of the smaller circle \( S \). Then \( \kappa_S = \frac{1}{r} \). Then we see the following right triangle:
Hence, \( \cos \theta = \frac{r}{R} = \frac{\kappa_S}{\kappa_G} \), meaning that
\[
\kappa_S = \frac{\kappa_G}{\cos \theta}.
\]
\[\square\]

8. Compute the curvature \( \kappa_0 \) at the origin of this curve, \( \alpha_0(t) = (t, 0, f(t, 0)) \), and show that
\[
\kappa_0 = \frac{|\alpha_0' \times \alpha_0''|}{|\alpha_0'|^3} = |(0, -f_{xx}, 0)| = |f_{xx}|.
\]

Proof. First, note that
\[
\alpha_0' = (1, 0, f_x) \quad \alpha_0'' = (0, 0, f_{xx}).
\]
Note that, since the surface is tangent to the \( xy \)-plane at the origin, there is no change in the \( z \)-direction at the origin, so \( f_x = 0 \) at the origin. Hence,
\[
|\alpha_0'|^3 = 1
\]
and
\[
\alpha_0' \times \alpha_0'' = (0, -f_{xx}, 0),
\]
so
\[
\kappa_0 = \frac{|\alpha_0' \times \alpha_0''|}{|\alpha_0'|^3} = |(0, -f_{xx}, 0)| = |f_{xx}|.
\]
\[\square\]

9. Compute the curvature \( \kappa_\alpha \) at the origin of the more general curve
\[
\alpha(t) = (t, g(t), f(t, g(t))),
\]
and show that
\[
\kappa_\alpha = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = |(0, -f_{xx}, g'')| = \sqrt{f_{xx}^2 + (g'')^2}.
\]

Proof. First, note that
\[
\alpha' = (1, g', f_x + f_y g')
\]
and
\[
\alpha'' = (0, g'', f_{xx} + f_y g'' + f_{xy} g' + f_{yy} (g')^2).
\]
Now, since the surface is tangent to the \( xy \)-plane, so is the curve \( g(t) \), so \( g'(0) = 0 \) and \( f_x = f_y = 0 \) at the origin. Hence,
\[
|\alpha'|^3 = 1
\]
and
\[
\alpha' \times \alpha'' = (g'(f_{xx} + f_y g'') + f_{xy} g'' + f_{yy} (g')^2) - g''(f_x + f_y g') - (f_{xx} + f_y g'' + f_{xy} g' + f_{yy} (g')^2, g'') = (0, -f_{xx}, g'').
\]
Hence,
\[ \kappa_\alpha = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = |(0, -f_{xx}, g'')| = \sqrt{f_{xx}^2 + (g'')^2}. \]

\[ 10. \]

Show that \( \kappa_\alpha = \kappa_0/|\cos \theta| \).

Proof. Note that the tangent vector to \( \alpha \) at the origin is given by \( \alpha' = (1, g', f_x + f_y g') \). Therefore, the principal normal is given by
\[ N_\alpha = \frac{\alpha''}{|\alpha''|} = \frac{1}{\sqrt{(g'')^2 + (f_{xx} + f_y g' + f_{yy} (g')^2)^2}} (0, g'', f_{xx} + f_y g' + f_{yy} (g')^2) \]
\[ = \frac{1}{\kappa_\alpha} (0, g'', f_{xx}) \]
since \( g', f_x \) and \( f_y \) all vanish at the origin. Hence,
\[ |N||N_\alpha| \cos \theta = (N, N_\alpha) = \frac{f_{xx}}{\kappa_\alpha}. \]

Since \( |N| = |N_\alpha| = 1 \), this implies that \( \cos \theta = \frac{f_{xx}}{\kappa_\alpha} \). Therefore,
\[ \frac{\kappa_0}{|\cos \theta|} = \frac{|f_{xx}|}{\kappa_\alpha} = \kappa_\alpha. \]

\[ 12. \]

Consider the curve \( \alpha(t) \) on \( S \) given by
\[ \alpha(t) = \left( t \cos \theta, t \sin \theta, \frac{1}{2} at^2 \cos^2 \theta + \frac{1}{2} bt^2 \sin^2 \theta \right). \]

Show that
\[ k_\theta = k_1 \cos^2 \theta + k_2 \sin^2 \theta, \]
which tells us how the normal curvature varies with direction.

Proof. First, note that
\[ \alpha' = (\cos \theta, \sin \theta, at^2 \cos^2 \theta + bt \sin^2 \theta) \]
and
\[ \alpha'' = (0, 0, a \cos^2 \theta + b \sin^2 \theta). \]

Since \( S \) is tangent to the \( xy \)-plane at the origin, the \( z \)-coordinate of \( \alpha' \) is zero; hence,
\[ |\alpha'|^3 = (\cos^2 \theta + \sin^2 \theta)^{3/2} = 1. \]
Also,
\[ \alpha' \times \alpha'' = (\sin \theta(a \cos^2 \theta + b \sin^2 \theta), - \cos \theta(a \cos^2 \theta + b \sin^2 \theta), 0), \]
so
\[ |\alpha' \times \alpha''| = \sqrt{(\sin^2 \theta + \cos^2 \theta)(a \cos^2 \theta + b \sin^2 \theta)^2} = a \cos^2 \theta + b \sin^2 \theta. \]

Therefore,
\[ k_\theta = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = \frac{a \cos^2 \theta + b \sin^2 \theta}{1} = k_1 \cos^2 \theta + k_2 \sin^2 \theta, \]
since \( k_1 = a \) and \( k_2 = b \).

See attached Maple sheets.

Let \( V \) be a small neighborhood of the point \( p \) on the regular surface \( S \), and \( N(V) \subset S^2 \) its image under the Gauss map \( N \). Show that the Gaussian curvature \( K \) of \( S \) at \( p \) is given by the limit
\[ \kappa = \lim_{V \to p} \frac{\text{area}(N(V))}{\text{area}(V)}. \]

Show that this generalizes an analogous result for the curvature of plane curves.

**Proof.** By definition, \( \text{Area}(V) = \int_V |X_u \times X_v|dA \). Now,
\[ \text{Area}(N(V)) = \int_{N(V)} |N_u \times N_v|dA = \int_{N(V)} |K(u, v)||X_u \times X_v|dA. \]

Now, as \( V \to p \), \( |K(u, v)| \) goes to the constant map \( K(p) \), the Gaussian curvature at \( p \), so we can pull it outside the integral. Hence,
\[ \lim_{V \to p} \frac{\text{Area}(N(V))}{\text{Area}(V)} = \lim_{V \to p} \frac{\int_{N(V)} |K||X_u \times X_v|dA}{\int_V |X_u \times X_v|dA} = K. \]