4.

Give $\text{Diff}(S^1)$ any reasonable topology, such as the compact-open topology, the $C^r$ topology or the $C^\infty$ topology. The orthogonal group $O(2)$ of rigid motions of a circle has two components, the rotations $SO(2)$ and the flips. Find an explicit strong deformation retraction of $\text{Diff}(S^1)$ to $O(2)$.

**Answer:** We may as well consider elements of $\text{Diff}(S^1)$ as functions of $\phi$, the angle of elevation above the $x$-axis. Now, if $f \in \text{Diff}(S^1)$, define $H : \text{Diff}(S^1) \times [0,1] \to \text{Diff}(S^1)$ by

$$H(f,t)(\phi) = (1-t)f(\phi) + tA_f(\phi)$$

where $A_f \in O(2)$ is chosen such that $A_f(0) = f(0)$ and, if $f$ is orientation-preserving, so is $A_f$ (i.e. $A_f$ is a rotation) and if $f$ is orientation-reversing, so is $A_f$ (i.e. $A_f$ is a flip). Note that if $f \in \text{Diff}(S^1)$,

$$H(f,0)(\phi) = f(\phi)$$

$$H(f,1)(\phi) = A_f(\phi)$$

for all $\phi$ and, if $B \in O(2)$, $A_B = B$, so $H(B,t)(\phi) = B(\phi)$ for all $\phi$. Since $H(f,t) \in \text{Diff}(S^1)$ for all $t$, we need only show that $H$ is continuous to demonstrate that $H$ is a strong deformation retraction of $\text{Diff}(S^1)$ onto $O(2)$.

We give $\text{Diff}(S^1)$ the compact-open topology; since addition and multiplication are continuous, we need only demonstrate that the choice of $A_f$ is continuous in this topology. We can denote $A_f = A(f)$ where $A : \text{Diff}(S^1) \to O(2)$ is as defined above. Let

$$V_{C,U} = \{f \in O(2) | C \text{ compact}, U \text{ open and } f(C) \subset U\}$$

be a basic open set in $O(2)$ where $C$ is a closed interval $[\phi_1,\psi_1]$ and $U$ is an open interval $(\phi_2,\psi_2)$. Then an orientation-preserving element in $V_{C,U}$ (i.e. a rotation) must rotate $\phi = 0$ to at least $\phi_2 - \phi_1$ and at most $\psi_2 - \psi_1$; that is, for all rotations $B \in V_{C,U}$, $B(0) \in (\phi_2 - \phi_1,\psi_2 - \psi_1)$. Hence, if $f \in A^{-1}(V_{C,U})$ is orientation-preserving, then $f(0) \in (\phi_2 - \phi_1,\psi_2 - \psi_1)$. In other words, if $O$ denotes the set of orientation-preserving elements of $\text{Diff}(S^1)$, then

$$A^{-1}(V_{C,U}) \cap O = W_{[0],(\phi_2-\phi_1,\psi_2-\psi_1)} \cap O,$$

where

$$W_{X,Y} = \{f \in \text{Diff}(S^1) | f(X) \subset Y\}.$$
Since \{0\} is compact and \((\phi_2 - \phi_1, \psi_2 - \psi_1)\) is open in \(S^1\), this set is open in \(Diff(S^1)\) provided we can show that \(O\) is open in \(Diff(S^1)\) (see below).

On the other hand, an orientation-reversing element of \(V_{C,U}\) must flip \(\phi = 0\) to at least \(\psi_2 - \phi_1\) and at most \(\phi_2 - \psi_1\); that is, for all flips \(B \in V_{C,U}\), \(B(0) \in (\psi_2 - \phi_1, \phi_2 - \psi_1)\). Hence, if \(f \in A^{-1}(V_{C,U})\) is orientation-reversing, then \(f(0) \in (\psi_2 - \phi_1, \phi_2 - \psi_1)\). In other words, if \(\overline{O}\) denotes the set of orientation-reversing elements of \(Diff(S^1)\), then
\[
A^{-1}(V_{C,U}) \cap \overline{O} = W_{\{0\},(\psi_2 - \phi_1, \phi_2 - \psi_1)} \cap \overline{O}.
\]
Again, this set is open in \(Diff(S^1)\) provided \(\overline{O}\) is open. Putting the above facts together, we see that
\[
A^{-1}(V_{C,U}) = (W_{\{0\},(\phi_2 - \phi_1, \psi_2 - \psi_1)} \cap O) \cup (W_{\{0\},(\psi_2 - \phi_1, \phi_2 - \psi_1)} \cap \overline{O}),
\]
which is open in \(Diff(S^1)\) if \(O\) and \(\overline{O}\) are. Before we prove that \(O\) and \(\overline{O}\) are open, note that this will suffice to complete the proof. Since the open sets in \(S^1\) are generated by the open intervals, when considering open sets \(V_{C,U}\) in \(O(2)\), it suffices to consider the case where \(U\) is an interval. On the other hand, if \(K\) is any compact set in \(S^1\), let \(C_K\) be the interval determined by the “largest” and “smallest” elements of \(K\) (that is, \(C_K\) is a [possibly degenerate] interval of shortest length containing \(K\); note that there may be more than one such: in such a case, simply choose one). Then, since we’re only considering rigid motions of \(S^1\), \(V_{K,U} = V_{C_K,U}\). Therefore, we can restrict our attention to the case where \(K\) is an interval (again, possibly a degenerate interval consisting of a single point).

Now, we turn to demonstrating that \(O\) and \(\overline{O}\) are both open in \(Diff(S^1)\). Since \(O \cap \overline{O} = \emptyset\) and \(O \cup \overline{O} = Diff(S^1)\), this will mean that \(O\) and \(\overline{O}\) are precisely the connected components of \(Diff(S^1)\). Let \(\phi_1 < \phi_2 < \phi_3\). For each \(f \in Diff(S^1)\), choose neighborhoods \(U_{f,i}\) about each of the \(\phi_i\) such that the \(U_{f,i}\) are disjoint and the \(f(U_{f,i})\) are disjoint. Note that, since \(f\) is a diffeomorphism, \(f(U_{f,i})\) is open for all \(i\). Hence
\[
W_f := W_{\{\phi_1\},f(U_{f,1})} \cap W_{\{\phi_2\},f(U_{f,2})} \cap W_{\{\phi_3\},f(U_{f,3})}
\]
is open. Furthermore, \(f \in W_f\). Also, if \(f\) is orientation-preserving, then \(W_f\) contains only orientation-preserving maps, and if \(f\) is orientation-reversing, then \(W_f\) contains only orientation-reversing maps. That is, \(W_f\) is an open neighborhood containing \(f\) and entirely contained in either \(O\) or \(\overline{O}\), depending on whether \(f\) is orientation-preserving or -reversing. Since our choice of \(f \in Diff(S^1)\) was arbitrary, this implies that both \(O\) and \(\overline{O}\) are open, completing the proof.

\[\diamondsuit\]

5.

Prove that the subgroup \(\mathcal{D}\) is contractible.
Proof. Let \( f \in \mathcal{D} \). Then, since \( f \) fixes an interval about \( \phi = 0 \), \( f \) is an orientation-preserving diffeomorphism. Furthermore, \( f(0) = 0 \), so, using the notation from problem 4 above, \( H(f, 1) = \text{Id} \), the identity map. Since our choice of \( f \in \mathcal{D} \) was arbitrary, we see that \( H|_{\mathcal{D}} \) homotopes all of \( \mathcal{D} \) to the identity. Thus, \( \mathcal{D} \) is contractible. \( \square \)

7

(1) Compute \( I_1(h_{t,d}) \) explicitly for all \( h_{t,d} \in B^2 \).

Answer: Recall that

\[
h_{t,d}(\phi) = \begin{cases} 
\frac{\pi}{2} + (1 + d) \left( \phi - \frac{\pi}{2} \right) - t & \text{for } \phi \in D_2 \\
\frac{3\pi}{2} + (1 - d) \left( \phi - \frac{3\pi}{2} \right) + t & \text{for } \phi \in D_4
\end{cases}
\]

Also,

\[
I_1(h) = \int_0^{2\pi} gh^{-1}(\phi)N(\phi)d\phi
= \int_0^{2\pi} mN(\phi)d\phi + \int_{h(D_2 \cup D_4)} (M - m)N(\phi)d\phi
= \int_{h(D_2 \cup D_4)} (M - m)N(\phi)d\phi,
\]

so

\[
I_1(h_{t,d}) = \int_{h_{t,d}(D_2 \cup D_4)} (M - m)N(\phi)d\phi
= (M - m) \int_{h_{t,d}(7\pi/16)}^{h_{t,d}(9\pi/16)} (\cos \phi, \sin \phi)d\phi + (M - m) \int_{h_{t,d}(25\pi/16)}^{h_{t,d}(23\pi/16)} (\cos \phi, \sin \phi)d\phi
= (M - m) \left[(\sin \phi, -\cos \phi)_{7\pi/16}^{9\pi/16 + d\pi/16} - (\sin \phi, -\cos \phi)_{25\pi/16}^{23\pi/16 + d\pi/16 + t}\right]
= (M - m) \left[(\sin(\phi + d\pi/16 + t) - \sin(\phi))_{7\pi/16}^{9\pi/16 + d\pi/16} + (\sin(\phi + d\pi/16 + t) - \sin(\phi))_{23\pi/16}^{25\pi/16 + d\pi/16 + t}\right]
= \left[(\sin(\phi + d\pi/16 + t) - \sin(\phi))_{7\pi/16}^{9\pi/16 + d\pi/16} + (\sin(\phi + d\pi/16 + t) - \sin(\phi))_{23\pi/16}^{25\pi/16 + d\pi/16 + t}\right]
\]

See attached Maple sheet for a (slight) simplification of this expression.

♣

(2) Show that the arrows along the outside of the “proof in one picture” are a fair portrayal of the values of \( I_1(h_{t,d}) \in \Sigma^1 \).
**Answer:** Using the above,
\[
I_1(h_{3\pi/16,0}) \approx (M - m)(0.434, 0)
\]
\[
I_1(h_{3\pi/16,1/2}) \approx (M - m)(0.431, 0.320)
\]
\[
I_1(h_{0,1/2}) \approx (M - m)(0.385)
\]
\[
I_1(h_{-3\pi/16,1/2}) \approx (M - m)(-0.431, 0.320)
\]
\[
I_1(h_{-3\pi/16,0}) \approx (M - m)(-0.434, 0)
\]
\[
I_1(h_{3\pi/16,-1/2}) \approx (M - m)(-0.431, -0.320)
\]
\[
I_1(h_{0,-1/2}) \approx (M - m)(0, -0.385)
\]
\[
I_1(h_{3\pi/16,-1/2}) \approx (M - m)(0.431, -0.320)
\]

So the arrows on the “proof in one picture” really are realistic.

(3) Show that $I_1$ embeds $B^2$ into the plane $\mathbb{R}^2$ and takes the center of $B^2$ to the origin. Conclude that $I_1(\Sigma^1)$ loops once around the origin.

**Proof.** I can’t figure out how to simplify $I_1(H_{t,d})$ enough to make any headway showing that it’s injective (which would be enough to show that it’s an embedding, given that $I_1$ is a continuous map from a compact set into $\mathbb{R}^2$). Note that when $(t, d) = (0, 0),
\[
I_1(h_{0,0}) = (M - m)(\sin 9\pi/16 - \sin 7\pi/16 + \sin 25\pi/16 - \sin 23\pi/16,
- \cos 9\pi/16 + \cos 7\pi/16 - \cos 25\pi/16 + \cos 23\pi/16)
= (0, 0).
\]

Assuming $I_1$ embeds $B^2$ into the plane and since we know $I_1$ maps $(0, 0)$ to the origin, the boundary $\Sigma^1$ of $B^2$ wraps exactly once about the origin. □

(4) Find explicitly the points on $I_1(\Sigma^1)$ which are closest to the origin.

**Answer:** We restrict $I_1$ to each of the four sides of $\Sigma^1$ and look for critical points (see attached Maple sheets for computations). The critical points are at
\[
(t, d) = (\pm 3\pi/16, 0), (0, \pm 1/2).
\]

Checking each critical point and the corners of $\Sigma^1$, we see that the distance to the origin is minimized at $(t, d) = (0, \pm 1/2), with the distance given by $\approx 0.38(M - m)$.

(5) Show how to choose $\epsilon$ small enough so that $I(\Sigma^1)$ also loops once around the origin.

**Answer:** Note that so long as $\|I(h_{t,d}) - I_1(h_{t,d})\| < 0.38(M - m)$, then $I(\Sigma^1)$ must loop once around the origin. Note that $f$ is positive and bounded above on $S^1$; let $B$ be an upper bound on $f$ which is
at least 1. Now, let $\epsilon = \frac{M-m}{25B}$. Recall that the measure of $A$ is less than $4\epsilon$. Then

$$\|I(h_{t,d}) - I_1(h_{t,d})\| = \left\| \int_0^{2\pi} fh^{-1}_{t,d}(\phi) N(\phi) d\phi - \int_0^{2\pi} gh^{-1}_{t,d}(\phi) N(\phi) d\phi \right\|$$

$$= \left\| \int_0^{2\pi} (fh^{-1}_{t,d} - gh^{-1}_{t,d})(\phi) N(\phi) d\phi \right\|$$

$$= \left\| \int_{h_{t,d}(E_1 \cup D_2 \cup E_3 \cup D_4)} (fh^{-1}_{t,d}(\phi) - gh^{-1}_{t,d}(\phi)) N(\phi) d\phi \right\|$$

$$+ \left\| \int_{h_{t,d}(A)} (fh^{-1}_{t,d}(\phi) - gh^{-1}_{t,d}(\phi)) N(\phi) d\phi \right\|$$

$$\leq \left\| \int_{h_{t,d}(E_1 \cup D_2 \cup E_3 \cup D_4)} (fh^{-1}_{t,d}(\phi) - gh^{-1}_{t,d}(\phi)) N(\phi) d\phi \right\|$$

$$+ \left\| \int_{h_{t,d}(A)} (fh^{-1}_{t,d}(\phi) - gh^{-1}_{t,d}(\phi)) N(\phi) d\phi \right\|$$

$$\leq \left\| \int_{h_{t,d}(E_1 \cup D_2 \cup E_3 \cup D_4)} \epsilon N(\phi) d\phi \right\| + \left\| \int_{h_{t,d}(A)} fh^{-1}_{t,d}(\phi) N(\phi) d\phi \right\|$$

since both $f$ and $g$ are positive. Now, since $f$ is bounded by $B$, this implies that

$$\|I(h_{t,d}) - I_1(h_{t,d})\| \leq 2\pi \epsilon + B \left\| \int_{h_{t,d}(A)} N(\phi) d\phi \right\|$$

$$\leq 2\pi \epsilon + B \text{measure}(A)$$

Since $h_{t,d}$ stretches by a factor of at most $3/2$, $\text{measure}(A) \leq 3\epsilon/2$, so

$$\|I(h_{t,d}) - I_1(h_{t,d})\| \leq 2\pi \epsilon + 3B\epsilon/2 = \frac{2\pi(M - m)}{25B} + \frac{3(M - m)}{50}$$

$$\leq (M - m) \left( \frac{2\pi}{25} + \frac{3}{50} \right)$$

$$\approx 0.311(M - m)$$

$$< 0.38(M - m),$$

since $B \geq 1$. Hence, $I(\Sigma^1)$ does indeed wrap once around the origin.

(6) Conclude that there must be a root of the equation

$$I(h) = \int_0^{2\pi} fh^{-1}(\phi) N(\phi) d\phi = 0$$

somewhere inside that 2-cell $B^2$. 
Proof. As we showed in (5) above, \( I(h_{t,d}) \) has winding number about the origin of 1 where \( (t, d) \in \Sigma^1 \). On the other hand, for \(|(h_{t,d})| = 0\), \( I(t, d) = (0, 0) \) is a single point and so certainly has winding number 0 about the origin. Now, since \( B^2 \) is contractible, we can continuously deform it to the point \((0, 0)\); composing with \( I \) yields another continuous deformation. Suppose, for the sake of contradiction, that \( I(h_{t,d}) \) is never 0. Then the image under \( I \) of the contraction of \( \Sigma^1 \) can never cross the origin, so its winding number about the origin can never change. But this implies that 1 = 0, which is obviously bad. Thus, we conclude that \( I(h) = 0 \) does indeed have a solution; in particular, it has a solution contained in \( \mathcal{D} \). \( \square \)

B

Let \( C \) be a smooth simple closed curve in 3-space. Let \( r > 0 \) be a real number, and let \( N(C, r) \) denote the set of points in 3-space whose distance from \( C \) is at most \( r \). If \( r \) is sufficiently small, then these disks will not intersect one another, and hence \( N(C, r) \) will be homeomorphic to the product of \( C \) and any one of these disks, and will look like a curvy, perhaps knotted, solid donut. Assume \( r \) is this small, and prove that the volume of \( N(C, r) \) equals the length of \( C \) times the area, \( \pi r^2 \), of any one of the disks.

Proof. First, suppose \( C \) has non-vanishing curvature everywhere. Then the Frenet frame is defined everywhere on \( C \). We may as well assume \( C \) is parametrized by arc length. Let \( L \) be the length of \( C \). Consider a vertical cylinder \( Cyl \) with radius \( r \) and height \( L \); that is,
\[
Cyl := \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 \leq r, z \in [0, L]\}.
\]
Now, using the notation that \( C(s) = (C_1(s), C_2(s), C_3(s)) \), the normal vector \( N(s) = (N_1(s), N_2(s), N_3(s)) \) and the binormal vector \( B(s) = (B_1(s), B_2(s), B_3(s)) \), define the function \( f : Cyl \to N(C, r) \) by
\[
f(x, y, z) = C(z) + xN(z) + yB(z);
\]
then \( f(x, y, z) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)) \) where
\[
f_1(x, y, z) = C_1(z) + xN_1(z) + yB_1(z) \\
f_2(x, y, z) = C_2(z) + xN_2(z) + yB_2(z) \\
f_3(x, y, z) = C_3(z) + xN_3(z) + yB_3(z).
\]
Thus, the Jacobian of \( f \) is given by
\[
J(f) = \begin{pmatrix}
N_1(z) & B_1(z) & C_1'(z) + xN_1'(z) + yB_1'(z) \\
N_2(z) & B_2(z) & C_2'(z) + xN_2'(z) + yB_2'(z) \\
N_3(z) & B_3(z) & C_3'(z) + xN_3'(z) + yB_3'(z)
\end{pmatrix}
\]
Now, since \( z \) is equal to arc length, the first column of this matrix is just \( N(z) \), the second is \( B(z) \) and the third is
\[
C'(z) + xN'(z) + yB'(z) = T(z) + x(-\kappa(z)T(z) + \tau(z)B(z)) + y(-\tau(z)N(z))
\]
by the Frenet equations. Hence,
\[
\det J(f) = \langle N(z) \times B(z), T(z) - \kappa(z)xT(z) + \tau(z)xB(z) - \tau(z)yN(z) \rangle \\
= \langle T(z), T(z) - \kappa(z)xT(z) + \tau(z)xB(z) - \tau(z)yN(z) \rangle \\
= 1 - x\kappa(z)
\]
Now, the volume of \( N(C, r) \) is given by
\[
\int \int \int_{Cyl} \det J(f) = \int_0^L \int_0^r \int_0^{2\pi} (1 - \rho\kappa(z) \cos \theta) \rho d\theta d\rho dz \\
= \int_0^L \int_0^r 2\pi \rho d\rho dz \\
= \int_0^L \pi r^2 dz \\
= L\pi r^2,
\]
as expected.

Now, if \( C \) has zero curvature somewhere, we’ll split up the computation of the volume. Over those regions where the curvature is non-zero, we’ll integrate the Jacobian as above. Over those regions with zero curvature, \( C \) is cylindrical, so the volume of each such region is simply given by \( L_i \pi r^2 \) where \( L_i \) is the length of each region \( C_i \) with zero curvature. Then, simply adding the volumes for each of these regions yields a total volume of \( L\pi r^2 \). \( \square \)

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