Let $X$ be a normed linear space and $Y$ a linear subspace. The set of all continuous linear functionals on $X$ that are zero on $Y$ is called the annihilator of $Y$ and denoted by $Y^\perp$. Show that $Y^\perp$ is a closed linear subspace of the dual space, $X'$, of $X$. Show that $\overline{Y}^\perp = Y^\perp$.

Proof. To see that $Y^\perp$ is a linear subspace, suppose $l_1, l_2 \in Y^\perp$. Then, if $y \in Y$

$$(l_1 + l_2)(y) = l_1(y) + l_2(y) = 0$$

so $l_1 + l_2 \in Y^\perp$. Also, if $c \in \mathbb{R}$, $(cl_1)(x) = cl_1(x) = 0$, so $cl_1 \in Y^\perp$. To see that $Y^\perp$ is closed, suppose $l_k \in Y^\perp$ and $l_k \to L$. Let $\epsilon > 0$. Then, for any $y \in Y$, there exists $N \in \mathbb{N}$ such that, for $n > N$,

$$|l_n(y) - L(y)| < \epsilon/2.$$ 

Hence,

$$|L(y)| = |0 - L(y)| = |l_n(y) - L(y)| < \epsilon/2.$$ 

Since $|L(y)|$ is smaller than any positive number, it must be the case that $L(y) = 0$, meaning $L \in Y^\perp$, so $Y^\perp$ is closed.

Now, if $\overline{Y}$ signifies the completion of $Y$, then it is clear that

$$\overline{Y}^\perp \subseteq Y^\perp,$$

since $Y \subseteq \overline{Y}$. On the other hand, suppose $L \in Y^\perp$. Let $y \in \overline{Y}$. Then there exists a sequence $y_j \in Y$ such that $y_j \to y$. Since $L$ is continuous,

$$y_j \to y \Rightarrow L(y_j) \to L(y).$$

Since $y_j \in Y$, $L(y_j) = 0$ for all $j \in \mathbb{N}$, so their limit $L(y) = 0$. Since our choice of $y$ was arbitrary, we see that $L \in \overline{Y}^\perp$, so $Y^\perp \subseteq \overline{Y}^\perp$.

Since containment goes in both directions, we conclude that $\overline{Y}^\perp = Y^\perp$. $\square$

Consider $C[0,1]$ with the uniform norm and let $K \subset C[0,1]$ be the set of functions with the property that $\int_0^{1/2} f(t)dt - \int_{1/2}^1 f(t)dt = 1$. Show that $K$ is a closed convex set not containing the origin but $K$ has no point closest to the origin.
Proof. Suppose there exists a sequence \( f_n \in K \) such that the \( f_n \) converge to \( f \) in \( C[0,1] \). Now, since this convergence is uniform, we are justified in switching limits and integrals:

\[
\int_{1/2}^{1} f(t) dt - \int_{1/2}^{1} f(t) dt = \int_{1/2}^{1} \lim_{n \to \infty} f_n(t) dt - \int_{1/2}^{1} \lim_{n \to \infty} f_n(t) dt = \lim_{n \to \infty} \int_{1/2}^{1} f_n(t) dt - \lim_{n \to \infty} \int_{1/2}^{1} f_n(t) dt = 1.
\]

Hence, \( f \in K \), so \( K \) is closed. Also, for \( 0 \leq \alpha \leq 1 \) and \( f, g \in K \),

\[
\int_{1/2}^{1} \alpha f(t) dt - \int_{1/2}^{1} f(t) dt = \alpha \left( \int_{1/2}^{1} f(t) dt - \int_{1/2}^{1} f(t) dt \right) + \left( \int_{1/2}^{1} g(t) dt - \int_{1/2}^{1} g(t) dt \right) - \left( \int_{1/2}^{1} g(t) dt - \int_{1/2}^{1} g(t) dt \right) = \alpha + 1 - \alpha = 1.
\]

Hence, \( K \) is convex.

Finally, consider the sequence \( f_n \) as drawn below:

Then, certainly, \( f_n \in K \). Now, for each \( n \),

\[
||f_n||_{\text{unif}} = \max |f_n(x)| = n \frac{n}{n-2}.
\]

Hence, \( \inf ||f_n||_{\text{unif}} = \lim_{n \to \infty} \frac{n}{n-2} = 1 \). However, suppose there exists \( f \in K \) such that \( ||f||_{\text{unif}} \leq 1 \). Then

\[
\int_{0}^{1/2} f(t) dt - \int_{1/2}^{1} f(t) dt \leq \int_{0}^{1/2} 1 dt - \int_{0}^{1/2} -1 dt = 1/2 + 1/2 = 1.
\]

Since \( f \in K \), this inequality must actually be an equality, meaning that

\[
f(t) = \begin{cases} 
1 & t \leq 1/2 \\
-1 & t > 1/2 
\end{cases}
\]

However, this is not a continuous function, so \( f \notin K \). Hence

\[
\inf_{f \in K} ||f||_{\text{unif}} = 1
\]

but this infimum is never achieved, so there is no point in \( K \) closest to the origin. \( \square \)

Let \( K \subset L^1(0,1) \) be the set of all \( f \) with \( \int_{0}^{1} f(t) dt = 1 \). Show that \( K \) is a closed convex set (of \( L^1(0,1) \)) that does not contain the origin but there are infinitely many points in \( K \) that minimize the distance to the origin.
Proof. To see that $K$ is closed, suppose that $f$ is the limit of a sequence $f_k \in K$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that, if $n > N$,

$$||f - f_n||_1 < \epsilon.$$ 

Now,

$$\left| \int_0^1 f(t)dt - 1 \right| = \left| \int_0^1 f(t)dt - \int_0^1 f_n(t)dt \right|
= \left| \int_0^1 (f(t) - f_n(t))dt \right|
\leq \int_0^1 |f(t) - f_n(t)|dt
= ||f - f_n||_1
< \epsilon.$$ 

Since $\left| \int_0^1 f(t)dt - 1 \right|$ is arbitrarily small, we conclude that $\int_0^1 f(t)dt = 1$, which is to say that $f \in K$. Hence, $K$ is closed. To see that $K$ is convex, suppose $f, g \in K$. Then, for $0 \leq \alpha \leq 1$

$$\int_0^1 (\alpha f(t) + (1 - \alpha)g(t))dt
= \int_0^1 (\alpha f(t) + g(t) - \alpha g(t))dt
= \alpha \int_0^1 f(t)dt + \int_0^1 g(t)dt - \alpha \int_0^1 g(t)dt
= \alpha + 1 - \alpha
= 1.$$ 

Hence, $K$ is convex. Certainly the origin is not in $K$, since $\int_0^1 0 dt = 0 \neq 1$. However, for any $f \in K$,

$$||f - 0||_1 = ||f||_1 = \int_0^1 |f(t)|dt \geq \int_0^1 f(t)dt = 1$$

Furthermore, if $g \in K$ such that $f > 0$,

$$||g - 0||_1 = ||g||_1 = \int_0^1 |g(t)|dt = \int_0^1 g(t)dt = 1$$

There are infinitely many such $g$, so there are infinitely many points in $K$ that minimize the distance to the origin. 

Consider points $x = (x_1, x_2)$ in $\mathbb{R}^2$ with the norm $||x|| = |x_1| + |x_2|$. Let the subspace $V$ be the $x_1$ axis and define the linear functional $\ell$ by $\ell((x_1, 0)) = x_1$.

a) Show that $\ell$ is a bounded linear functional on $V$.

Proof. Let $(x_1, 0), (y_1, 0) \in V$. Then

$$\ell(c(x_1, 0) + d(y_1, 0)) = \ell((cx_1 + dy_1, 0)) = cx_1 + dy_1 = c\ell((x_1, 0)) + d\ell((y_1, 0))$$

for any $c, d \in \mathbb{R}$, so $\ell$ is certainly linear on $V$. Now, to see that $\ell$ bounded, we need only note that, if $x = (x_1, 0) \in V$

$$|\ell(x)| = |\ell((x_1, 0))| = |x_1| = |x_1| + 0 = ||x||$$

so $|\ell(x)| \leq ||x||$. 

Furthermore, for any \( y \in \mathbb{R}^2 \), we see that the Hahn-Banach extension is far from unique. We have infinitely many such, we see that the Hahn-Banach extension is far from unique. Let \( c, d \in \mathbb{R} \). Then

\[
\ell'(f) = \int_0^{1/2} f(t) \, dt.
\]

Then \( \ell' \) is certainly linear and bounded on \( V \). To see this, suppose \( f, g \in V \), \( c, d \in \mathbb{R} \). Then

\[
\ell'(cf + dg) = \int_0^{1/2} cf(t) + dg(t) \, dt = c \int_0^{1/2} f(t) \, dt + d \int_0^{1/2} g(t) \, dt = c\ell'(f) + d\ell'(g).
\]

For boundedness, if \( f \in V \),

\[
|\ell'(f)| = \left| \int_0^{1/2} f(t) \, dt \right| \leq \int_0^{1/2} |f(t)| \, dt = \int_0^1 |f(t)| \, dt = ||f||_1.
\]

Now, any functional of the form

\[
\mathcal{L}(f) = \int_0^{1/2} f(t) \, dt + c \int_{1/2}^1 f(t) \, dt
\]

where \( 0 \leq c \leq 1 \) will be an extension of \( \ell' \) to all of \( L^1(0,1) \). Since there are infinitely many such, we see that the Hahn-Banach extension is far from unique.

\[
5
\]

Let \( \ell^1, \ell^\infty \) denote the Banach spaces of all complex sequences \( x = \{x_j\} \) with the usual norms \( ||x||_1 = \sum |x_j| < \infty \) and \( ||x||_\infty = \sup |x_j| < \infty \) and let \( c_0 \) be the subspace of sequences in \( \ell^\infty \) that converge to zero.

a) If \( z \in \ell^1 \) show that \( \ell(x) := \sum z_j x_j \) defines a bounded linear functional on \( c_0 \) with \( ||\ell|| = ||z||_1 \).

Proof. Let \( x \in \ell^\infty \). Then

\[
|\ell(x)| = \left| \sum z_j x_j \right| \leq \sum |z_j x_j| \leq \sum |z_j| ||x||_\infty = \left( \sum |z_j| \right) ||x||_\infty = ||z||_1 ||x||_\infty,
\]

\[
\Box
\]
so \( \ell \) is bounded. Furthermore, if \( x, y \in \ell^\infty, c, d \in \mathbb{R} \),

\[
\ell(cx + dy) = \sum z_j(cx_j + dy_j) = c \sum z_j x_j + d \sum z_j y_j = c\ell(x) + d\ell(y),
\]

so \( \ell \) is linear. To see that \( ||\ell|| = ||z||_1 \), let \( \epsilon > 0 \). Then, since \( \sum |z_j| \) converges, we know there exists \( N \in \mathbb{N} \) such that, if \( n > N \),

\[
\sum_{j=n+1}^\infty |z_j| < \epsilon/2.
\]

Let \( x = (x_j)_{j=1}^\infty \), where \( x_j = z_j \) for \( j \leq N \), \( x_j = 0 \) for \( j > N \). Then \( x \in c_0 \) and

\[
||\ell(x) - ||z||_1|| = \left| \sum z_j x_j - \sum z_j \right| = \left| \sum_{N+1}^\infty |z_j| \right| < \epsilon/2.
\]

Hence, we see that \( ||\ell|| = ||z||_1 \). \( \square \)

b) Every bounded linear functional on \( c_0 \) is as above. In other words, \( \ell^1 \) is the dual space of \( c_0 \).

Proof. Suppose \( \ell \) is a bounded linear functional on \( c_0 \). Denote by \( e_j \) the vector with a 1 in the \( j \)th position and zeros everywhere else. Let \( x \in c_0 \). Then

\[
|\ell(x)| = |\ell(\sum_0^\infty x_je_j)| = |\sum_0^\infty x_j \ell(e_j)| \\
\leq \sum_0^\infty |x_j||\ell(e_j)| \\
\leq \sum_0^\infty ||x||_\infty|\ell(e_j)| \\
= ||x||_\infty \sum_0^\infty |\ell(e_j)| \\
= c||x||_\infty
\]

where \( \sum |\ell(e_j)| = c < \infty \), so \( \{\ell(e_j)\} \subseteq \ell^1 \). Hence, we see that the dual space \( c_0' \subseteq \ell^1 \). Since we showed the reverse containment in part (a), we conclude that \( c_0' = \ell^1 \). \( \square \)

c) Show that \( \ell^\infty \) is the dual space of \( \ell^1 \).

Proof. Let \( x \in \ell^\infty \). Define \( \ell(z) := \sum x_j z_j \). Then, for \( y, z \in \ell^1 \), \( c, d \in \mathbb{C} \),

\[
\ell(cy + dz) = \sum x_j (cy_j + dz_j) = c \sum x_j y_j + d \sum x_j z_j = c\ell(y) + d\ell(z),
\]

so \( \ell \) is linear on \( \ell^1 \). Furthermore, if \( z \in \ell^1 \),

\[
|\ell(z)| = \left| \sum x_j z_j \right| \leq \sum |x_j||z_j| \leq \sum ||x||_\infty|z_j| = ||x||_\infty \left( \sum |z_j| \right) = ||x||_\infty||z||_1,
\]

so \( \ell \) is bounded. Furthermore, just as in part (a), \( ||\ell|| = ||x||_\infty \).

Now, if \( \ell \) is a bounded linear functional on \( \ell^1 \) and \( x \in \ell^1 \), then

\[
|\ell(x)| = |\ell(\sum x_je_j)| \leq \sum |x_j||\ell(e_j)| \leq ||\ell|| \sum |x_j| = ||\ell|| ||x||_1
\]

Since we’ve shown a correspondance between the linear functionals on \( \ell^1 \) and the elements of \( \ell^\infty \), we conclude that \( \ell^\infty \) is the dual space of \( \ell^1 \). \( \square \)
d) Show that \( \ell^1 \) is contained in the dual space \((\ell^\infty)'\) of \( \ell^\infty \), but is not all of \((\ell^\infty)'\) since there are elements in \((\ell^\infty)'\) that are zero on all of \( c_0 \).

6

a) With the positive linear functional \( \ell(f) := \int_\mathbb{R} f(t) dt \) on \( C_c(\mathbb{R}) \) show that the set \( \{ x \in \mathbb{R} : 0 \leq x < 1 \} \) is measurable.

Proof. Let \( A = \{ x \in \mathbb{R} : 0 \leq x < 1 \} \). Let \( \epsilon > 0 \). Define \( \Omega := (-\epsilon/2, 1) \).

Then \( \Omega \Delta A = (\Omega \cup A) \setminus (\Omega \cap A) = (-\epsilon/2, 1) \setminus [0, 1) = (-\epsilon/2, 0) \).

Then \( M^*(\Omega \Delta A) = \inf_{\Omega \Delta A \subseteq O, O \text{ open}} \text{Vol}(O) = \text{Vol}((-\epsilon/2, 0)) = \sup_{g \in C_c((-\epsilon/2, 0))} (I(g)) \) where \( 0 \leq g(x) \leq 1 \). Suppose \( f(x) \) is 1 on \((-\epsilon/2, 0)\) and zero everywhere else. Then

\[
M^*(\Omega \Delta A) = \sup_{g \in C_c((-\epsilon/2, 0))} (\ell(g)) \leq \ell(f) = \int_{-\epsilon/2}^0 1 dx = -\epsilon/2 < \epsilon.
\]

However, this is just the definition of measurable, so we see that \( A \) is measurable. \( \square \)

b) With the positive linear functional \( \ell(f) := \int_\mathbb{R}^2 f(x, y) dx dy \) on \( C_c(\mathbb{R}^2) \) show that the set \( \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 < y < 1\} \) is measurable.

Proof. Let \( B = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 < y < 1\} \) and let \( \epsilon > 0 \). Define \( \Omega := [0, 1] \times [0, 1] \).

Then \( \Omega \Delta B = ([0, 1] \times [0, 1]) \cap ([0, 1] \times (0, 1)) = (0 \times (0, 1)) \cup (1 \times (0, 1)) \).

Now,

\[
M^*(\Omega \Delta B) = \inf_{\Omega \Delta B \subseteq O} \text{Vol}(O)
\]

where \( O \) is open. Now, consider the sequence of open sets \( O_n = ((-1/n, 1/n) \times (0, 1)) \cup ((1 - 1/n, 1 + 1/n) \times (0, 1)) \). Then

\[
\text{Vol}(O_n) = \sup_{g \in C_d(O_n), 0 \leq g(x) \leq 1} (\ell(g)) \leq f_{-1/n}^{1/n} 1 dy dx + f_{1-1/n}^{1+1/n} 1 dy dx = 2/n + 2/n = 4/n.
\]

Certainly, the sequence \( 4/n \to 0 \), so we see that

\[
M^*(\Omega \Delta B) \leq \inf_{n \in \mathbb{N}} \text{Vol}(O_n) = \inf_{n \in \mathbb{N}} \frac{4}{n} = 0.
\]

Since \( M^*(S)/geq0 \) for any set \( S \), we conclude that

\[
M^*(\Omega \Delta B) = 0.
\]

Therefore, \( B \) is measurable, \( \square \)
c) With the positive linear functional $\ell(f) := f(0) + 2f(1)$ on $C_c(\mathbb{R})$ discuss which sets are measurable. Can you compute their measures?

**Answer:** Suppose $S$ is a set containing neither 0 nor 1. Then we can construct an open set $\Omega$ containing $S$ such that $\Omega$ contains neither 0 nor 1. If $f \in C_c(\Omega)$, then $f(0) = f(1) = 0$, meaning $\ell(f) = f(0) + 2f(1) = 0$. Hence

$$M^*(S) = \inf_{S \subset O, O \text{ open}} \text{Vol}(O) \leq \text{Vol}(\Omega) = \sup_{g \in C_c(\Omega), 0 \leq g(x) \leq 1} (\ell(g)) = 0.$$ 

Therefore, $S$ has measure 0.

On the other hand, if $S'$ is a set containing 0 but not 1, then we can construct an open set $\Omega'$ containing $S'$ such that $\Omega'$ does not contain 1. If $f \in C_c(O)$ where $O$ is an open set not containing 1, then $f(1) = 0$, so $\ell(f) = f(0) + 2f(1) = f(0)$. Furthermore, if $O$ is an open set containing zero, then there exists a function $h_O \in C_c(O)$ such that $h_O(0) = 1$. Thus,

$$M^*(S') = \inf_{S' \subset O, O \text{ open}} \text{Vol}(O) \leq \inf_{S' \subset O, O \text{ open}} \left( \sup_{g \in C_c(O), 0 \leq g(x) \leq 1} (\ell(g)) \right) \leq \inf_{S' \subset O, O \text{ open}} (\ell(h_O)) = 1.$$ 

Therefore, $S'$ has measure 0.

A parallel argument demonstrates that if $S''$ contains 1 but not 0, then the measure of $S''$ is 2.

Finally, if $S_1$ contains both 0 and 1, then

$$M^*(S_1) = \inf_{S_1 \subset O, O \text{ open}} \text{Vol}(O).$$

If $Q$ is any open set containing $S_1$, there exists $f_Q \in C_c(Q)$ such that $f_Q(0) = f_Q(1) = 1$ and $0 \leq f(x) \leq 1$. Hence, $\ell(f_Q) = f_Q(0) + 2f_Q(1) = 3$, so

$$\text{Vol}(Q) = \sup_{g \in C_c(Q), 0 \leq g(x) \leq 1} (\ell(g)) \geq 3.$$ 

If we let $g_Q$ denote the function that is 1 on $Q$ and zero outside, then it is also clear that

$$\text{Vol}(Q) = \sup_{g \in C_c(Q), 0 \leq g(x) \leq 1} (\ell(g)) \leq (\ell(g_Q)) = 3.$$ 

Therefore, for any open $Q$ containing $S_1$, $\text{Vol}(Q) = 3$. Therefore,

$$M^*(S_1) = \inf_{S_1 \subset O, O \text{ open}} \text{Vol}(O) = 3,$$

so $S_1$ has measure 3.
In a Hilbert space $H$, a map $U : H \to H$ is called **unitary** if
(i). \( \langle Ux, Uy \rangle = \langle x, y \rangle \) for all $x, y$ in $H$.
(ii). $U$ is onto.
Let $Q := \{ x \in H : Ux = x \} = \ker(I - U)$ of all points left fixed by $U$.

a) Show that in finite dimensions property (ii) is a consequence of property (i).

**Proof.** Let $H$ be finite dimensional and let $U$ be unitary. Suppose $x, y \in H$ such that $Ux = Uy$. Then $0 = Ux - Uy = U(x - y)$, which is to say that $0 = \langle U(x - y), U(x - y) \rangle = \langle x - y, x - y \rangle \Rightarrow x - y = 0$.
Hence, $x = y$, so $U$ is injective. Since $U : H \to H$ is injective, it must be surjective, since $H$ is finite-dimensional. \( \square \)

b) In the Hilbert space $\ell^2$, show that the right shift
$S : (x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots)$ has property (i) but not property (ii).

**Proof.** Let $x, y \in \ell^2$. Then $x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots)$ for $x_i, y_i \in \mathbb{C}$. Then
\[
\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i} = 0 + \sum_{i=1}^{\infty} x_i \overline{y_i} = \langle Sx, Sy \rangle,
\]
so $S$ fulfills property (i). However, There is no $z \in \ell^2$ such that $S z = (1, 0, 0, \ldots)$,
so $S$ is not unitary. \( \square \)

c) View functions on the circle $S^1$ as functions periodic with period $2\pi$.
If $H = L^2(S^1)$ and $\alpha$ is real, show that the map $L : f(x) \mapsto f(x + 2\pi \alpha)$ is unitary.

**Proof.** Suppose $f \in L^2(S^1)$. Then
\[
L(f(x - 2\pi \alpha)) = f(x - 2\pi \alpha + 2\pi \alpha) = f(x),
\]
so $L$ is surjective.
Now, let $f, g \in L^2(S^1)$. Then
\[
\langle Lf, Lg \rangle = \int_{S^1} (Lf)(Lg) = \int_0^{2\pi} f(x + 2\pi \alpha)g(x + 2\pi \alpha) dx = \int_{2\pi + 2\pi \alpha}^{2\pi} f(u)g(u) du = \int_{S^1} (f)(g) = \langle f, g \rangle
\]
where we let $u = x + 2\pi \alpha$. Therefore, $L$ is unitary. \( \square \)

d) If $\alpha$ is irrational, find the set $Q$.

**Answer:** If $\alpha$ is irrational, then, if $f \in Q$,
\[
f(x) = Lf = L^m f = f(x + 2\pi m \alpha)
\]
for all $m \in \mathbb{Z}$. Since $Q \subseteq L^2(S^1)$, $f$ must also be $2\pi$-periodic. Hence, either $2\pi m\alpha = 2\pi$ for some integer $m$ or $2\pi \alpha = n2\pi$ for some integer $n$. Dividing by $2\pi$ on both sides of each, we see that

$$m\alpha = 1 \text{ or } \alpha = \frac{n}{2\pi}.$$

Since $\alpha$ is irrational, neither of these is possible, so we conclude that $Q$ is empty if $\alpha$ is irrational.

\begin{itemize}
  \item[e)] If $\alpha$ is rational, find the set $Q$.
  \textbf{Answer:} If $\alpha$ is rational, then, again, if $f \in Q$,
  $$f(x) = Lf = L^m f = f(x + 2\pi m\alpha)$$
  for all $m \in \mathbb{Z}$. Since $\alpha$ is rational, $\alpha = p/q$ for $p \in \mathbb{Z}$, $q \in \mathbb{Z}^*$. Also, if $f \in Q$, $f$ must be $2\pi$-periodic. Hence, the elements of $Q$ will be precisely those functions which are $\frac{2\pi n q}{m}$-periodic, where $n \in \mathbb{Z}^*$.
\end{itemize}

\begin{itemize}
  \item[f)] If $U$ is unitary, show that $U^* = U^{-1}$ and hence that $Ux = x$ if and only if $U^*x = x$. Consequently $\ker(I - U) = \ker(I - U)^*$.\end{itemize}

\textbf{Proof.} Let $x, y \in H$. By definition,

$$\langle x, Uy \rangle = \langle U^*x, y \rangle = \langle UU^*x, Uy \rangle.$$  

Hence,

$$0 = \langle x, Uy \rangle - \langle UU^*x, Uy \rangle = \langle x - UU^*x, Uy \rangle.$$  

Since our choice of $y$ was arbitrary, we see that it must be the case that $x - UU^*x = 0$, or

$$x = UU^*x \Rightarrow U^{-1}x = U^*x.$$  

Since our choice of $x$ was arbitrary, this means that $U^{-1} = U^*$.

Hence, $Ux = x$ if and only if $x = U^{-1}x = U^*x$, and so $\ker(I - U) = \ker(I - U)^*$.

\begin{itemize}
  \item[g)] Using $(\text{Image}L)^\perp = \ker L^*$ for any bounded linear operator $L$ in a Hilbert space, conclude that $H = \ker(I - U) \oplus \overline{\text{Image}(I - U)}$.
\end{itemize}

\textbf{Proof.} From problem 1 above, we know that

$$\overline{\text{Image}(I - U)} = (\text{Image}(I - U))^\perp = \ker(I - U)^* = \ker(I - U),$$

where we use the given fact to get the second equality and part (g) to get the third. Now, since we are in a Hilbert space,

$$(\ker(I - U))^\perp = (\overline{\text{Image}(I - U)})^\perp = \overline{\text{Image}(I - U)}.$$  

Applying this result to the conclusion proved in problem 5 of last week’s homework, we see that

$$H = \ker(I - U) \oplus (\ker(I - U))^\perp = \ker(I - U) \oplus \overline{\text{Image}(I - U)}.$$  

\end{itemize}
8. (Continuation)

von Neumann’s mean ergodic theorem states that for any \( f \in H \), one has

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{0}^{N-1} U^n x = P x
\]

where \( P \) is the orthogonal projection onto the set \( Q := \{ x \in H : U x = x \} = \text{ker}(I - U) \). Equation (1) is clearly true for all \( x \in Q = \ker(I - U) \).

a) Prove that for all \( z \in \text{Image}(I - U) \), that is \( z = (I - U)x \) for some \( x \in H \), then

\[
\left\| \frac{1}{N} \sum_{0}^{N-1} U^n z \right\| = \left\| \frac{1}{N} (x - U^N x) \right\| \to 0.
\]

Proof. We see immediately that

\[
\sum_{0}^{N-1} U^n z = \sum_{0}^{N-1} U^n (I - U)x
\]

\[
= \sum_{0}^{N-1} U^n x - U^{n+1} x
\]

\[
= (x - Ux) + (Ux - U^2x) + \ldots + (U^{N-1}x - U^N x)
\]

\[
= x - U^N x.
\]

Hence,

\[
\left\| \frac{1}{N} \sum_{0}^{N-1} U^n z \right\| = \left\| \frac{1}{N} (x - U^N x) \right\|. 
\]

\( \square \)

b) Conclude that (2) holds for all \( z \in \text{Image}(I - U) \).

Proof. First, note that, for all \( x \in H \),

\[
\|Ux\|^2 = \langle Ux, Ux \rangle = \langle x, x \rangle = \|x\|^2,
\]

so \( \|Ux\| = \|x\| \).

Now, let \( z \in \text{Image}(I - U) \). Then there exists a sequence \( \{z_j\} \in \text{Image}(I - U) \) such that \( z_j \to z \). Let \( \epsilon > 0 \). Then there exists \( M \in \mathbb{N} \) such that, if \( k > M \),

\[
\|z - z_k\| < \epsilon/2.
\]

Also, from part (a), we know that for all \( k \) there exists \( N \in \mathbb{N} \) such that, if \( n > N \),

\[
\left\| \frac{1}{n} \sum_{0}^{n-1} U^m z_k \right\| < \epsilon/2.
\]
Let \( k > M, n > N \). Therefore,

\[
\left| \frac{1}{n} \sum_{0}^{n-1} U^m z \right| = \left| \frac{1}{n} \sum_{0}^{n-1} U^m (z - z_k) + \frac{1}{n} \sum_{0}^{n-1} U^m z_k \right|
\]

\[
\leq \left| \frac{1}{n} \sum_{0}^{n-1} U^m (z - z_k) \right| + \left| \frac{1}{n} \sum_{0}^{n-1} U^m z_k \right|
\]

\[
\leq \frac{1}{n} \sum_{0}^{n-1} ||U^m (z - z_k)|| + \left| \frac{1}{n} \sum_{0}^{n-1} U^m z_k \right|
\]

\[
= \frac{1}{n} \sum_{0}^{n-1} ||z - z_k|| + \left| \frac{1}{n} \sum_{0}^{n-1} U^m z_k \right|
\]

\[
< \epsilon/2 + \epsilon/2
\]

\[
= \epsilon.
\]

Hence, for all \( z \in \text{Image}(I - U) \),

\[
\left| \frac{1}{N} \sum_{0}^{N-1} U^n z \right| \to 0.
\]

\(\square\)

c) Use the last part of the previous problem to show that \( Pz = 0 \) if and only if \( z \in \text{Image}(I - U) \). Thus (2) proves (1) for all \( x \) that satisfy \( Px = 0 \) and thus for all \( x \) in the Hilbert space.

**Proof.** Certainly, as we’ve shown in (b) above, if \( z \in \text{Image}(I - U) \), then

\[
\lim N \to \infty \sum_{0}^{N-1} U^n z = 0.
\]

Furthermore, as we saw in 7(g), if \( z \in \text{Image}(I - U) \), then \( z \in (\ker(I - U)^*)^\perp \), meaning \( Pz = 0 \). Hence,

\[
z \in \text{Image}(I - U).
\]

On the other hand, if \( Pz = 0 \), then \( z \in (\ker(I - U)^*)^\perp \) and, therefore, by 7(g), \( z \in \text{Image}(I - U) \). By (b), then,

\[
Pz = 0 = \lim_{N \to \infty} \sum_{0}^{N-1} U^n z.
\]

\(\square\)

9

Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and \( a(x) > 0 \) and \( f(x) \) be smooth functions periodic with period \( 2\pi \). Let \( H^1(S^1) \) be the completion of smooth \( 2\pi \) periodic functions on the circle \( S^1 \) in the norm

\[
||\phi||_{H^1}^2 := \int_{S^1} (|\phi'|^2 + \phi^2) \, dx.
\]
Call \( u \in H^1(S^1) \) a weak solution of \( -u'' + a(x)u = f \) on \( S^1 \) if
\[
\int_{S^1} [u'v' + a(x)uv]dx = \int_{S^1} fvdx \quad \text{for all} \quad v \in H^1(S^1).
\]

a) (uniqueness) Show that the equation \(-u'' + a(x)u = f\) on \( S^1 \) has at most one weak solution.

Proof. Suppose there exist two weak solutions, \( u_1 \) and \( u_2 \). Then
\[
0 = \int_{S^1} fvdx - \int_{S^1} fdx = \int_{S^1} [u_1'v' + a(x)u_1v]dx - \int_{S^1} [u_2'v' + a(x)u_2v]dx
\]
\[
= \int_{S^1} [(u_1' - u_2')(v' + a(x)(u_1 - u_2))v]dx
\]
for all \( v \in H^1(S^1) \). Particularly, when \( v = u_1 - u_2 \), \( v' = u_1' - u_2' \), so
\[
\int_{S^1} [(u_1' - u_2')^2 + a(x)(u_1 - u_2)]dx = 0.
\]
Since \( a(x) > 0 \), this means that \( u_1 - u_2 = 0 \) or \( u_1 = u_2 \), so any weak solution must be unique. \( \square \)

b) (existence) Show that the equation \(-u'' + a(x)u = f\) on \( S^1 \) has a solution.

Proof. Define the norm \( || \cdot ||^2_H \) such that, for \( v \in H^1(S^1) \),
\[
||v||^2_H = \int_{S^1} [(v')^2 + a(x)v^2]dx.
\]
Define the functional \( \ell(v) := \int_{S^1} fv \). Then \( \ell \) is certainly linear and
\[
|\ell(v)| = \left| \int_{S^1} fv \right| \leq ||f||_{L^2}||v||_{L^2} \leq c||f||_{L^2}||v||^2_H
\]
by the Holder inequality and since this norm is equivalent to the \( L^2 \) norm, so \( \ell \) is a bounded linear functional on the Hilbert space. Hence, by the Riesz-Frechet Representation Theorem, there exists \( u \in H^1(S^1) \) such that \( \ell(v) = \langle v, u \rangle \). Therefore
\[
\int_{S^1} fv = \ell(v) = \langle v, u \rangle = \int_{S^1} [v'u' + a(x)vu]dx,
\]
so \( u \) is a weak solution. \( \square \)

c) Show that if \( b(x) \) is a smooth function on \( S^1 \) with \( |b(x)| \) sufficiently small, then the equation \(-u'' + b(x)u' + a(x)u = f\) on \( S^1 \) has exactly one weak solution.

Proof. A weak solution \( u \) to this equation must satisfy
\[
\int_{S^1} [u'v' - bv'u + a(x)vu]dx = \int_{S^1} fv
\]
for all \( v \in H^1(S^1) \).
To show uniqueness of weak solutions, suppose $u_1$ and $u_2$ are weak solutions. Then, if we let $w = u_1 - u_2$,

$$0 = \int_{S^1} f v - \int_{S^1} f v = \int_{S^1} [u_1'v' - b(x)v'u_1 + a(x)v u_1] dx - \int_{S^1} [u_2'v' - b(x)v'u_2 + a(x)v u_2] dx$$

$$= \int_{S^1} [u_1'v' - b(x)v'u_2 + a(x)v u_2 - u_1'v' + b(x)v'u_1 - a(x)v u_1] dx$$

for all $v$. Particularly, when $v = w$

$$0 = \int_{S^1} [(w')^2 - b(x)w' w + a(x)w^2] dx.$$

If we integrate the second term by parts, we will get

$$0 = \int_{S^1} [(w')^2 B(x)(w')^2 + a(x)w^2] dx = \int_{S^1} [(1 + B(x))(w')^2 + a(x)w^2] dx$$

where $B(x) = \int_0^x b(t) dt$. Now, if $|b(x)|$ is sufficiently small for all $x \in S^1$, then $|B(x)| > -1$, which in turn implies that $(1 + B(x)) > 0$. Since $a(x) > 0$, all terms in the above integral are nonnegative, so we conclude that $w = 0$, meaning $u_1 = u_2$. Therefore, any weak solution to this equation is unique.

To prove existence, we now need to define another new norm, $|| \cdot ||^2_H$, where, for $v \in H^1(S^1)$,

$$||v||^2_H = \int_{S^1} [(v')^2 - b(x)v' v + a(x)v^2] dx.$$

Let $\ell(v) = \int_{S^1} f v$. Then, again, $\ell$ is linear and, using Holder,

$$|\ell(v)| = \left| \int_{S^1} f v \right| \leq ||f||_{L^2} ||v||_{L^2} \leq c ||f||_{L^2} ||v||^2_H$$

since this norm is equivalent to the $L^2$ norm. So $\ell$ is a bounded linear functional on the Hilbert space. Now, in order to construct a weak solution $u$, we need to introduce a bilinear form $B(\cdot, \cdot)$, where, for $v, w \in H^1(S^1)$,

$$B(v, w) = \int_{S^1} [v'w' - b(x)v' w + a(x)v w] dx.$$

Now, we need to check the following:

(i) $B$ is bounded

(ii) There exists $b > 0$ such that $|B(y, y)| \geq b||y||^2$ for all $y \in H^1(S^1)$.

To see (i), let $v, w \in H^1(S^1)$. Then

$$|B(v, w)| = \left| \int_{S^1} [v'w' - b(x)v' w + a(x)v w] dx \right|$$

$$\leq \int_{S^1} |v'w' dx| + |\int_{S^1} b(x)v' w dx| + |\int_{S^1} a(x)v w dx|$$

$$= \int_{S^1} a(x)v w dx$$

$$\leq ||v||_{L^2} ||w||_{L^2}$$

$$\leq c ||v||^2_H ||w||^2_H.$$

To prove (ii), we merely note that

$$|B(v, v)| = \left| \int_{S^1} [(v')^2 - b(x)v' v + a(x)v^2] dx \right| = ||v||_{L^2} = ||v||^2,$$
so $|B(v, v)| \geq ||v||^2$ for all $v$. Therefore, by the Lax-Milgram Lemma, there exists $u$ such that $\ell(v) = B(v, u)$. This means that

$$\int_{S^1} fv = \ell(v) = B(v, u) = \int_{S^1} [v'u' - b(x)v'u + a(x)vu]dx,$$

meaning $u$ is a weak solution of the given equation. \qed

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