3.2.19

Prove that if \( N \) is a normal subgroup of the finite group \( G \) and \( (|N|, |G : N|) = 1 \) then \( N \) is the unique subgroup of \( G \) of order \(|N|\).

**Proof.** Let \( H \leq G \) such that \(|H| = |N|\). By Proposition 13,

\[
|NH| = \frac{|N||H|}{|N \cap H|} = \frac{|N|^2}{|N \cap H|} = |N| \frac{|N|}{|N \cap H|}
\]

Since \( N \) is normal, \( NH \leq G \) by Corollary 15, so \(|NH| \) divides \(|G|\), or

\[
|G| = m|NH| = m|N| \frac{|N|}{|N \cap H|}.
\]

On the other hand,

\[
|G| = |G : N||N|,
\]

so

\[
|N||G : N| = |N|m\frac{|N|}{|N \cap H|} \quad \text{cancelling yields}
\]

\[
|G : N| = m\frac{|N|}{|N \cap H|}.
\]

Hence, \( \frac{|N|}{|N \cap H|} \) divides both \(|N|\) and \(|G : N|\) so, by hypothesis,

\[
\frac{|N|}{|N \cap H|} = 1
\]

Therefore, \(|N \cap H| = |N|\), which means, since \(|H| = |N|\), that \( H = N \). We conclude, then, that \( N \) is the unique subgroup of \( G \) with order \(|N|\). \( \square \)

3.4.2

Find all 3 composition series for \( Q_8 \) and all 7 composition series for \( D_8 \).

List the composition factors in each case.

**Answer:**

\[
1 \leq \{1, -1\} \leq \{1, -1, i, -i\} \leq Q_8
\]

\[
1 \leq \{1, -1\} \leq \{1, -1, j, -j\} \leq Q_8
\]

and

\[
1 \leq \{1, -1\} \leq \{1, -1, k, -k\} \leq Q_8
\]

where, in each case, \( N_{i+1}/N_i = \mathbb{Z}/2\mathbb{Z} \).

\[
1 \leq \langle s \rangle \leq \langle s, r^2 \rangle \leq D_8
\]

\[
1 \leq \langle r^2 \rangle \leq \langle s, r^2 \rangle \leq D_8
\]
1 ⊆ ⟨r²⟩ ⊆ ⟨r⟩ ⊆ D₈
1 ⊆ ⟨sr⟩ ⊆ ⟨sr, sr³⟩ ⊆ D₈
1 ⊆ ⟨sr³⟩ ⊆ ⟨sr, sr³⟩ ⊆ D₈
1 ⊆ ⟨sr²⟩ ⊆ ⟨s, sr²⟩ ⊆ D₈
and
1 ⊆ ⟨s⟩ ⊆ ⟨s, sr²⟩ ⊆ D₈
where, in each case, \( N_{i+1}/N_i \equiv \mathbb{Z}/2\mathbb{Z} \).

3.4.5

Prove that subgroups and quotient groups of a solvable group are solvable.

Proof. Let \( G \) be a solvable group with composition series
\[ 1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G. \]
Let \( H \leq G \). Consider \( H \cap G_i \) and \( H \cap G_{i+1} \). Each is clearly a group. If \( g \in H \cap G_i \) and \( h \in H \cap G_{i+1} \), then \( hgh^{-1} \in H \) since \( g, h \in H \). Also, \( hgh^{-1} \in G_i \), since \( G_i \trianglelefteq G_{i+1} \), so we can conclude that
\[ H \cap G_i \trianglelefteq H \cap G_{i+1}. \]
This also implies that \( H \cap G_{i+1} \leq N_G(H \cap G_i) \). Hence, we see that \( H \) is solvable, as we can construct the series
\[ 1 \trianglelefteq H \cap G_1 \trianglelefteq \ldots H \cap G_{n-1} \trianglelefteq H \cap G_n = H \cap G = H. \]
\[ \square \]

3.5.4

Show that \( S_n = \langle (12), (123 \ldots n) \rangle \) for all \( n \geq 2 \).

Proof. We want to show that for any transposition \( (pq) \) where \( p < q \), \( (pq) \in \langle (12), (123 \ldots n) \rangle \). Now,
\[ (12 \ldots n)(12)(12 \ldots n)^{-1} = (12 \ldots n)(12)(1n \ldots 2) = (23) \]
and, in general,
\[ (12 \ldots n)(m(m+1))(12 \ldots n)^{-1} = ((m+1)(m+2)). \]
Furthermore, for a transposition \( (pq) \),
\[ (pq) = ((q-1)q) \ldots ((p+1)(p+2))(p(p+1))((p+1)(p+2)) \ldots ((q-1)q). \]
Since each term on the right is generated by \( (12) \) and \( (12 \ldots n) \), then so is \( (pq) \). Since our choice of transposition was arbitrary, we see that every transposition in \( S_n \) is in \( \langle (12), (12 \ldots n) \rangle \). Since, as we’ve seen, every element of \( S_n \) can be written as a product of transpositions, we see that, for all \( \sigma \in S_n, \sigma \in \langle (12), (12 \ldots n) \rangle \). Hence, \( S_n = \langle (12), (12 \ldots n) \rangle \). \[ \square \]
3.5.10

Find a composition series for $A_4$. Deduce that $A_4$ is solvable.

**Answer:**

$$1 \leq \langle (12)(34) \rangle \leq \langle (12)(34), (13)(24) \rangle \leq A_4$$

is a composition series of $A_4$. To see that $A_4$ is, in fact, solvable, it suffices to note that

$$\langle (12)(34) \rangle / 1 \simeq \mathbb{Z}/2\mathbb{Z}$$

$$\langle (12)(34), (13)(24) \rangle / \langle (12)(34) \rangle \simeq \mathbb{Z}/2\mathbb{Z}$$

and

$$A_4 / \langle (12)(34), (13)(24) \rangle \simeq \mathbb{Z}/3\mathbb{Z},$$

each of which is a simple abelian group.

1

Let $G$ be a group. The opposite group, $G^{\text{op}}$, is the group which is equal to $G$ as a set, whose group law $\mu'$ is defined by

$$\mu'(x, y) = \mu_G(y, x) \quad \forall x, y \in G^{\text{op}}.$$  

Prove that $G^{\text{op}}$ is isomorphic to $G$.

**Proof.** Define $\phi : G \to G^{\text{op}}$ by

$$\phi(x) = x^{-1}.$$  

This is well-defined since $x^{-1} \in G$ is equal to $x^{-1} \in G^{\text{op}}$. Then

$$\ker(\phi)\{x \in G|x^{-1} = 1\} = \{1\}$$

so $\phi$ is injective. Also, for any $x \in G^{\text{op}},$

$$\phi(x^{-1}) = (x^{-1})^{-1} = x,$$

so $\phi$ is surjective. Finally, for $x, y \in G,$

$$\phi(\mu(x, y)) = \mu(x, y)^{-1} = \mu(y^{-1}, x^{-1}) = \mu'(x^{-1}, y^{-1}) = \mu'(\phi(x), \phi(y)),$$

so $\phi$ is a homomorphism. Since $\phi$ is a bijective homomorphism, it is an isomorphism.
Two homomorphisms \( f_1, f_2 \) from a group \( G_1 \) to a group \( G_2 \) are *conjugate* if there exists an element \( g \in G_2 \) such that \( f_1(x) = g f_2(x) g^{-1} \) for all \( x \in G_1 \).

(a) Find all homomorphisms from \( S_3 \) to \( \mathbb{C}^\times \).

Let \( f : S_3 \to \mathbb{C}^\times \) be a homomorphism. Then \( \ker(f) \trianglelefteq S_3 \). We know, from the last homework, that \( \{1\}, \langle (12) \rangle, \langle (13) \rangle, \langle (23) \rangle, \langle (123) \rangle \), \( S_3 \) comprises the entire list of subgroups of \( S_3 \). If \( \ker(f) = \{1\} \), then \( f \) is a monomorphism, meaning \( S_3 \simeq f(S_3) \). However, \( S_3 \) is not abelian, whereas \( \mathbb{C}^\times \) is, so this is impossible. If \( \ker(f) = S_3 \), then \( f \) is just the trivial homomorphism.

Now, we know that \( f \) induces an isomorphism \( f' : S_3/\ker(f) \to f(S_3) \). If \( |\ker(f)| = 2 \), then

\[
S_3/\ker(f) \simeq \mathbb{Z}/3\mathbb{Z},
\]

meaning \( f(S_3) \) is a cyclic subgroup of \( \mathbb{C}^\times \) of order 3. The only such group is the group of 3rd roots of unity, \( \{1, e^{\frac{2\pi\sqrt{-1}}{3}}, e^{\frac{4\pi\sqrt{-1}}{3}}\} \). Hence, there are precisely two possibilities for \( f' \) given \( \ker(f) \). For example, if \( \ker(f) = \langle (12) \rangle \), then

\[
S_3/\ker(f) = S_3/\langle(12)\rangle = \{(12), (13)(12), (23)(12)\}.
\]

Then \( f'((12)) = 1 \), \( f'((13)(12)) = e^{\frac{2\pi\sqrt{-1}}{3}} \) or \( f'((13)(12)) = e^{\frac{4\pi\sqrt{-1}}{3}} \) and \( f'((23)(12)) \) is whatever remains. Hence, we deduce that

\[
\begin{align*}
f(1) &= f((12)) = 1, f((13)) = f((123)) = e^{\frac{2\pi\sqrt{-1}}{3}}, f((23)) = f((132)) = e^{\frac{4\pi\sqrt{-1}}{3}} \\
or\quad f(1) &= f((12)) = 1, f((13)) = f((123)) = e^{\frac{4\pi\sqrt{-1}}{3}}, f((23)) = f((132)) = e^{\frac{2\pi\sqrt{-1}}{3}}.
\end{align*}
\]

Similarly, if \( \ker(f) = \langle (13) \rangle \), then

\[
\begin{align*}
f(1) &= f((13)) = 1, f((12)) = f((123)) = e^{\frac{2\pi\sqrt{-1}}{3}}, f((23)) = f((132)) = e^{\frac{4\pi\sqrt{-1}}{3}} \\
or\quad f(1) &= f((13)) = 1, f((12)) = f((123)) = e^{\frac{4\pi\sqrt{-1}}{3}}, f((23)) = f((132)) = e^{\frac{2\pi\sqrt{-1}}{3}}
\end{align*}
\]

and if \( \ker(f) = \langle (23) \rangle \), then

\[
\begin{align*}
f(1) &= f((23)) = 1, f((12)) = f((123)) = e^{\frac{2\pi\sqrt{-1}}{3}}, f((13)) = f((132)) = e^{\frac{4\pi\sqrt{-1}}{3}} \\
or\quad f(1) &= f((23)) = 1, f((12)) = f((123)) = e^{\frac{4\pi\sqrt{-1}}{3}}, f((13)) = f((132)) = e^{\frac{2\pi\sqrt{-1}}{3}}.
\end{align*}
\]

Finally, if \( |\ker(f)| = 3 \), then \( \ker(f) = \langle (123) \rangle \), so

\[
S_3/\ker(f) \simeq \mathbb{Z}/2\mathbb{Z}.
\]

That is to say that \( f(S_3) \) is a cyclic group of order two in \( \mathbb{C}^\times \). The only such possibility is the group \( \{1, -1\} \). This means \( f((123)) = 1 \) and \( f((12)(123)) = -1 \). Specifically,

\[
f(1) = f((123)) = f((132)) = 1, f((12)) = f((13)) = f((23)) = -1.
\]
Therefore, the above constitute all possible homomorphisms from $S_3$ to $\mathbb{C}^\times$.

(b) Determine all homomorphisms from $S_3$ to $S_3$ up to conjugation.

Clearly the trivial homomorphism $f(x) = 1$ for all $x \in S_3$ is one such. Replacing $e^{2\pi \sqrt{-1}}$ with $(123)$ and $e^{4\pi \sqrt{-1}}$ with $(132)$ makes it clear that the only such homomorphisms having $\langle (12) \rangle$ as their kernel are $f$ and $g$ such that:

\[
f(1) = f((12)) = 1, f((13)) = f((123)) = (123), f((23)) = f((132)) = (132)\]

and

\[
g(1) = g((12)) = 1, g((13)) = g((123)) = (132), g((23)) = g((132)) = (123).
\]

However, $f(x) = (12) g(x)(12)$ for all $x \in S_3$, so these two homomorphisms are conjugate. A similar argument gives a single homomorphism (up to conjugation) for each kernel $\langle (13) \rangle$ and $\langle (23) \rangle$. However, homomorphisms with different kernels of degree two will not be conjugate, as can be seen simply by noting that there is no $x \in S_3$ such that $x(123)x^{-1} = 1$.

Now, if $f$ is a homomorphism with kernel $\langle (123) \rangle$, then, paralleling our arguments in the previous part, we see that either

\[
f(1) = f((123)) = f((132)) = 1, f((12)) = f((13)) = f((23)) = (12)
\]

or

\[
(1) = f((123)) = f((132)) = 1, f((12)) = f((13)) = f((23)) = (13)
\]

or

\[
(1) = f((123)) = f((132)) = 1, f((12)) = f((13)) = f((23)) = (23).
\]

If we call the first of these possibilities $f_1$, the second $f_2$ and the third $f_3$, then it is readily apparent that $f_1(x) = (23)f_2(x)(23), f_1(x) = (13)f_3(x)(13)$ and $f_2(x) = (12)f_3(x)(12)$ for all $x \in S_3$, so $f_1, f_2$ and $f_3$ are conjugate.

Finally, if the kernel of a homomorphism from $S_3$ to $S_3$ is trivial, then that homomorphism is, in fact, an automorphism. In the last homework, we saw that any automorphism $f$ of $S_3$ is of the form

\[
f(x) = axa^{-1}
\]

for some $a \in S_3$ and for all $x \in S_3$. Hence, if $f$ and $g$ are two automorphisms of $S_3$, then $f(x) = axa^{-1}$ and $g(x) = bxb^{-1}$ for some $a, b \in S_3$. However,

\[
g(x) = bxb^{-1} = b(a^{-1}a)x(a^{-1}a)b^{-1} = (ba^{-1})(axa^{-1})(ab^{-1}) = (ba^{-1})f(x)(ba^{-1})^{-1},
\]

so $f$ and $g$ are conjugate. Therefore, we conclude that, up to conjugation, there is exactly one homomorphism of $S_3$ into itself for each of the 6 possible kernels.

(c) Prove that any two injective homomorphisms from $S_3$ to $GL_2(\mathbb{R})$ are conjugate.
Proof. Let $f$ and $g$ be monomorphisms from $S_3$ to $GL_2(\mathbb{R})$. Then $f(S_3)$ and $g(S_3)$ are isomorphic to $S_3$. Consider the map $g^{-1} \circ f : S_3 \rightarrow S_3$. Note that $g^{-1}$ is an isomorphism. If $x, y \in S_3$ such that $(g^{-1} \circ f)(x) = (g^{-1} \circ f)(y)$, then

$$g^{-1}(f(x)) = g^{-1}(f(y))$$

so $f(x) = f(y)$, meaning $x = y$, so $g^{-1} \circ f$ is injective. Since $S_3$ is finite, $g^{-1} \circ f$ is clearly surjective. Also, if $x, y \in S_3$,

$$(g^{-1} \circ f)(xy) = g^{-1}(f(xy)) = g^{-1}(f(x)f(y)) = g^{-1}(f(x))g^{-1}(f(y)) = (g^{-1} \circ f)(x)(g^{-1} \circ f)(y),$$

so $g^{-1} \circ f$ is an automorphism. Hence, as shown in last week’s homework, there exists $\tau \in S_3$ such that

$$(g^{-1} \circ f)(x) = \tau x \tau^{-1}$$

for all $x \in S_3$. Now,

$$f(x) = ((g \circ g^{-1} \circ f)(x) = (g \circ (g^{-1} \circ f))(x) = g((g^{-1} \circ f)(x)) = g(\tau x \tau^{-1}) = g(\tau)g(x)g(\tau^{-1}),$$

so we see that $f$ and $g$ are conjugate. Since our choice of $f$ and $g$ was arbitrary, we conclude that any two monomorphisms from $S_3$ to $GL_2(\mathbb{R})$ are conjugate. \qed