Example: Is this matrix diagonalizable?

Problem: Let

$$A = \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}.$$  

Is matrix $A$ diagonalizable?

**Answer:** By Proposition 23.2, matrix $A$ is diagonalizable if and only if there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of $A$. So let’s find the eigenvalues and eigenspaces for matrix $A$.

By Proposition 23.1, $\lambda$ is an eigenvalue of $A$ precisely when $\det(\lambda I - A) = 0$. Note $\lambda I - A$:

$$\lambda I - A = \begin{bmatrix} \lambda - 6 & -3 & 8 \\ 0 & \lambda + 2 & 0 \\ -1 & 0 & \lambda + 3 \end{bmatrix}.$$  

To find $\det(\lambda I - A)$ let’s do cofactor expansion along the second row because it has many zeros$^1$. We get

$$\det(\lambda I - A) = -0 \begin{vmatrix} -3 & 8 \\ 0 & \lambda + 3 \end{vmatrix} + (\lambda + 2) \begin{vmatrix} \lambda - 6 & 8 \\ -1 & \lambda + 3 \end{vmatrix} - 0 \begin{vmatrix} \lambda - 6 & -3 \\ -1 & \lambda + 3 \end{vmatrix}$$

$$= (\lambda + 2) \begin{vmatrix} \lambda - 6 & 8 \\ -1 & \lambda + 3 \end{vmatrix} - (\lambda - 6)(\lambda + 3) - 8(-1)$$

$$= (\lambda + 2)(\lambda^2 - 3\lambda - 10)$$

$$= (\lambda + 2)(\lambda - 5).$$

Hence our eigenvalues are $\lambda = -2$ and $\lambda = 5$. Note that $\lambda = -2$ is a repeated root with multiplicity two.

We find the corresponding eigenspaces. We get that

$$E_{-2} = N \begin{pmatrix} -8 & -3 & 8 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \ldots = N \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \ldots = \text{span} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$E_5 = N \begin{pmatrix} -1 & -3 & 8 \\ 0 & 7 & 0 \\ -1 & 0 & 8 \end{pmatrix} = \ldots = N \begin{pmatrix} 1 & 0 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \ldots = \text{span} \begin{pmatrix} 8 \\ 0 \\ 1 \end{pmatrix}. $$

The ellipses show where I have omitted work that you should know how to do, namely putting a matrix in reduced row echelon form and writing a null space as a span.

We have found only two linearly independent eigenvectors for $A$, namely the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}.$$  

But any basis for $\mathbb{R}^3$ consists of three vectors. Therefore there is no eigenbasis for $A$, and so by Proposition 23.2 matrix $A$ is not diagonalizable.

**Remark:** The reason why matrix $A$ is not diagonalizable is because the dimension of $E_{-2}$ (which is 1) is smaller than the multiplicity of eigenvalue $\lambda = -2$ (which is 2).

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$^1$In section we did cofactor expansion along the first column, which also works, but makes the resulting cubic polynomial harder to factor.