The exercises are from *Foundations of Mathematical Analysis* by Richard Johnsonbaugh and W.E. Pfaffenberger.

17.4. Let \( \{a_n\} \) be a sequence with positive terms such that \( \lim_{n \to \infty} a_n = L > 0 \). Let \( x \) be a real number. Prove that \( \lim_{n \to \infty} a_n^x = L^x \).

**Solution.** Let \( \epsilon > 0 \). By Theorem 17.4, note that \( L < (L^x + \epsilon)^{1/x} \) and \( L > (L^x - \epsilon)^{1/x} \). Since \( \lim_{n \to \infty} a_n = L \), there exists some \( N \) such that \( n \geq N \) implies \( a_n < (L^x + \epsilon)^{1/x} \) and \( a_n > (L^x - \epsilon)^{1/x} \). Hence by Theorem 17.4, for \( n \geq N \) we have \( a_n^x < L^x + \epsilon \) and \( a_n^x > L^x - \epsilon \). This shows \( \lim_{n \to \infty} a_n^x = L^x \).

19.3. Let \( 0 \leq \alpha < 1 \), and let \( f \) be a function from \( \mathbb{R} \to \mathbb{R} \) which satisfies

\[
|f(x) - f(y)| \leq \alpha|x - y|
\]

for all \( x, y \in \mathbb{R} \). Let \( a_1 \in \mathbb{R} \), and let \( a_{n+1} = f(a_n) \) for \( n = 1, 2, \ldots \). Prove that \( \{a_n\} \) is a Cauchy sequence.

**Solution.** First we prove by induction on \( n \) that \( |a_{n+1} - a_n| \leq \alpha^{n-1}|a_2 - a_1| \) for all \( n \in \mathbb{N} \). The base case \( n = 1 \) is obvious. Assuming the formula is true when \( n = k \), we show it is true for \( n = k + 1 \):

\[
|a_{k+2} - a_{k+1}| = |f(a_{k+1}) - f(a_k)| \leq \alpha|a_{k+1} - a_k| \leq \alpha^{k-1}|a_2 - a_1| = \alpha^k|a_2 - a_1|
\]

Hence, by induction, this formula is true for all \( n \).

Note that if \( |a_2 - a_1| = 0 \), then \( a_n = a_1 \) for all \( n \), and so the sequence is clearly Cauchy. Hence we consider the case when \( |a_2 - a_1| \neq 0 \). Now, given any \( \epsilon > 0 \), pick \( N \) such that \( \alpha^{N-1} < \frac{\epsilon}{|a_2 - a_1|} \), which we can do because \( 0 \leq \alpha < 1 \implies \lim_{n \to \infty} \alpha^n = 0 \). Then, for any \( m, n \geq N \), with \( m \geq n \), we have

\[
|a_m - a_n| \leq \sum_{i=n}^{m-1} |a_{i+1} - a_i| \quad \text{by the triangle inequality}
\]

\[
\leq \sum_{i=n}^{m-1} \alpha^{i-1}|a_2 - a_1| \quad \text{by our formula above}
\]

\[
= |a_2 - a_1| \sum_{i=n}^{m-1} \alpha^{i-1}
\]

\[
= |a_2 - a_1| \sum_{i=N}^{\infty} \alpha^{i-1}
\]

\[
= |a_2 - a_1| \frac{\alpha^{N-1}}{1 - \alpha} \quad \text{by the formula for the sum of an infinite geometric series}
\]

\[
< \epsilon \quad \text{by our choice of} \ N.
\]

Hence \( \{a_n\} \) is a Cauchy sequence.
20.6. Compute \( \limsup_{n \to \infty} a_n \) and \( \liminf_{n \to \infty} a_n \), where \( a_n \) is ...

**Solution.**

(a) \( \frac{1}{n} \)

Since we know \( \lim_{n \to \infty} \frac{1}{n} = 0 \), we know \( \limsup_{n \to \infty} \frac{1}{n} = \liminf_{n \to \infty} \frac{1}{n} \) by Theorem 20.4.

(b) \( (1 + \frac{1}{n})^n \)

Since we know \( \lim_{n \to \infty} (1 + \frac{1}{n})^n = e \), we know \( \limsup_{n \to \infty} (1 + \frac{1}{n})^n = e = \liminf_{n \to \infty} (1 + \frac{1}{n})^n \) by Theorem 20.4.

(c) \( (-1)^n (1 - \frac{1}{n}) \)

Since \(-1 \leq (1 + \frac{1}{n})^n \leq 1\) for all \( n \), we have \( \liminf_{n \to \infty} (1 + \frac{1}{n})^n \geq -1 \) and \( \limsup_{n \to \infty} (1 + \frac{1}{n})^n \leq 1 \). Since the subsequence \( \{a_{2n-1}\} \) has limit \(-1\) and the subsequence \( \{a_{2n}\} \) has limit 1, we have \( \liminf_{n \to \infty} (1 + \frac{1}{n})^n \leq -1 \) and \( \limsup_{n \to \infty} (1 + \frac{1}{n})^n \geq 1 \). Hence \( \liminf_{n \to \infty} (1 + \frac{1}{n})^n = -1 \) and \( \limsup_{n \to \infty} (1 + \frac{1}{n})^n = 1 \).

20.7. Compute \( \limsup_{n \to \infty} a_n \) and \( \liminf_{n \to \infty} a_n \) and \( \mathcal{L}_a \), where \( a_1, a_2, \ldots \) is an enumeration of the rational numbers in the closed interval \([0, 1]\).

**Solution.** We show that \( \mathcal{L}_a = [0, 1] \). It is easy to show \( \mathcal{L}_a \subset [0, 1] \).

To show \( \mathcal{L}_a \supset [0, 1] \), let \( s \in [0, 1] \). First we consider the case \( s > 0 \). By Theorem, 17.1, there exists an increasing rational sequence \( \{r_n\} \) with limit \( s \). As \( s > 0 \), for \( n \) sufficiently large we have \( r_n \geq 0 \), so we may assume that \( r_n \geq 0 \) for all \( n \), hence \( r_n \in [0, 1] \) for all \( n \). By induction on \( n \), we define a sequence \( \{b_n\} \) which is a subsequence of both \( \{a_n\} \) and \( \{r_n\} \). For the base case, set \( b_1 = r_1 = a_k \) for some integer \( k \). For the inductive step, suppose we have defined \( b_1, \ldots, b_n \) and \( b_n = r_l = a_k \). Since \( a_1, a_2, \ldots \) is an enumeration of the rational numbers, and since the set \( \{r_{l+1}, r_{l+2}, \ldots\} \) is infinite but \( \{a_1, \ldots, a_k\} \) is finite, there exists some \( k' > k \) such that \( a_{k'} = r_{l'} \) for some \( l' > l \). Set \( b_{n+1} = r_{l'} = a_{k'} \). Note that \( \{b_n\} \) is a subsequence of both \( \{a_n\} \) and \( \{r_n\} \). Since \( \{b_n\} \) is a subsequence of \( \{r_n\} \), we have \( \lim_{n \to \infty} b_n = \lim_{n \to \infty} r_n = s \). Since \( \{b_n\} \) is a subsequence of \( \{a_n\} \), this shows \( s \in \mathcal{L}_a \). The case \( s = 0 \) is analogous. Hence \( \mathcal{L}_a \supset [0, 1] \), so \( \mathcal{L}_a = [0, 1] \).

Since \( \mathcal{L}_a = [0, 1] \), we have \( \limsup_{n \to \infty} a_n = \text{lub}(\mathcal{L}_a) = 1 \) and \( \liminf_{n \to \infty} a_n = \text{glb}(\mathcal{L}_a) = 0 \).

20.9. Let \( \{a_n\} \) be a bounded sequence such that every convergent subsequence of \( \{a_n\} \) has a limit \( L \). Prove that \( \lim_{n \to \infty} a_n = L \).

**Solution.**

Method 1: Note that \( \mathcal{L}_a = \{L\} \). Hence \( \limsup_{n \to \infty} a_n = \text{lub}(\mathcal{L}_a) = L = \text{glb}(\mathcal{L}_a) = \liminf_{n \to \infty} a_n \). So by Theorem 20.4, \( \lim_{n \to \infty} a_n = L \).

Method 2: Suppose for a contradiction that \( \{a_n\} \) does not have limit \( L \). Hence there exists an \( \epsilon > 0 \) such that for any integer \( N \), there exists some \( n > N \) with \( |a_n - L| > \epsilon \). This allows us to define \( n_1 < n_2 < n_3 \ldots \) such that \( |a_{n_i} - L| > \epsilon \) for all \( i \). Since \( \{a_{n_i}\} \) is a bounded sequence, by Bolzano-Weierstrass it has a convergent subsequence, which clearly does not converge to \( L \). This is a contradiction, and so it must be that \( \lim_{n \to \infty} a_n = L \).

20.13. Let \( \{a_n\} \) and \( \{b_n\} \) be sequences such that \( \{a_n\} \) is convergent and \( \{b_n\} \) is bounded. Prove that

\[
\limsup_{n \to \infty} (a_n + b_n) = \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n
\]

and

\[
\liminf_{n \to \infty} (a_n + b_n) = \liminf_{n \to \infty} a_n + \liminf_{n \to \infty} b_n
\]
\textit{Solution.} Since \( \{a_n\} \) is convergent, it is bounded. So Theorem 20.6 gives us
\[
\limsup_{n \to \infty}(a_n + b_n) \leq \limsup_{n \to \infty}a_n + \limsup_{n \to \infty}b_n
\]
Let \( \lim_{n \to \infty}a_n = L \). Note if \( l \in L_b \), then \( l = \lim_{k \to \infty}b_{n_k} \) for some \( n_k \). Hence
\[
\lim_{k \to \infty}(a_{n_k} + b_{n_k}) = \lim_{k \to \infty}a_{n_k} + \lim_{k \to \infty}b_{n_k} = L + l
\]
so \( L + l \in L_{a+b} \). This shows that
\[
\limsup_{n \to \infty}(a_n + b_n) = \text{hup} L_{a+b} \geq L + \text{hup} L_b = \limsup_{n \to \infty}a_n + \limsup_{n \to \infty}b_n
\]
We’ve shown
\[
\limsup_{n \to \infty}(a_n + b_n) = \limsup_{n \to \infty}a_n + \limsup_{n \to \infty}b_n
\]
The proof that
\[
\liminf_{n \to \infty}(a_n + b_n) = \liminf_{n \to \infty}a_n + \liminf_{n \to \infty}b_n
\]
is analogous.

20.20. Let \( \{a_n\} \) be a sequence of positive numbers such that \( \lim_{n \to \infty}a_n = L \). Prove that \( \lim_{n \to \infty}(a_1a_2 \cdots a_n)^{1/n} = L \).

\textit{Solution.} Let \( \epsilon > 0 \). Then there exists some \( N \) such that \( n \geq N \) implies \( a_n \leq L + \epsilon \). Note we have
\[
(a_1 \cdots a_{N+m})^{1/(N+m)} = (a_1 \cdots a_N)^{1/(N+m)}(a_{N+1} \cdots a_{N+m})^{1/(N+m)} \leq (a_1 \cdots a_N)^{1/(N+m)}(L + \epsilon)^{m/(N+m)} = (a_1 \cdots a_N(L + \epsilon)^{-N})^{1/(N+m)}(L + \epsilon)
\]
Hence we have
\[
\limsup_{n \to \infty}(a_1a_2 \cdots a_n)^{1/n} = \limsup_{m \to \infty}(a_1 \cdots a_{N+m})^{1/(N+m)} \leq \limsup_{m \to \infty}(a_1 \cdots a_N(L + \epsilon)^{-N})^{1/(N+m)}(L + \epsilon) \quad \text{by Theorem 20.5}
\]
\[
= L + \epsilon \quad \text{by Theorem 16.4}
\]
Hence \( \limsup_{n \to \infty}(a_1a_2 \cdots a_n)^{1/n} \leq L + \epsilon \) for all \( \epsilon > 0 \), so \( \limsup_{n \to \infty}(a_1a_2 \cdots a_n)^{1/n} \leq L \). Analogously, one can show that \( \liminf_{n \to \infty}(a_1a_2 \cdots a_n)^{1/n} \geq L \). Hence by Theorem 20.2 and 20.4, we have \( \lim_{n \to \infty}(a_1a_2 \cdots a_n)^{1/n} = L \).

21.2. Let \( A_n = \text{lub}\{a_n, a_{n+1}, \ldots\} \) and \( B_n = \text{glb}\{a_n, a_{n+1}, \ldots\} \) for \( n = 1, 2, \ldots \). Compute \( A_n, B_n, \lim_{n \to \infty} A_n, \) and \( \lim_{n \to \infty} B_n \), where \( a_n = \)

\textit{Solution.}

(a) \( (-1)^n \)

Clearly \( A_n = 1 \) and \( B_n = -1 \) so \( \lim_{n \to \infty} A_n = 1 \) and \( \lim_{n \to \infty} B_n = -1 \).

(b) \( \frac{1}{n} \)

Clearly \( A_n = \frac{1}{n} \) and \( B_n = 0 \) so \( \lim_{n \to \infty} A_n = 0 \) and \( \lim_{n \to \infty} B_n = 0 \).

(c) \( (1 + \frac{1}{n})^n \)

By Theorem 16.6 the sequence \( \{(1 + \frac{1}{n})^n\} \) is increasing and convergent with limit \( e \). So \( A_n = e \) and \( B_n = (1 + \frac{1}{n})^n \) so \( \lim_{n \to \infty} A_n = e \) and \( \lim_{n \to \infty} B_n = e \).

(d) \( \frac{(-1)^n}{n} \)
22.4. Prove that the series \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \) converges and find its sum.

**Solution.** We use partial fractions to rewrite the terms. We try

\[
\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \implies 1 = (n+1)A + nB = A + n(A+B) \implies A = 1 \text{ and } B = -1.
\]

Indeed, we have \( \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \). So the \( n \)-th partial sum is

\[
s_n = a_1 + a_2 + \cdots + a_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}
\]

This is an example of a **telescoping** series. Since

\[
\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right) = 1,
\]

we have that \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \) converges, and \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \).

22.5. Give an example of divergent series \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) such that \( \sum_{n=1}^{\infty} (a_n + b_n) \) converges.

**Solution.** Let \( a_n = 1 \) and \( b_n = -1 \) for all \( n \). Then \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) are geometric series with \( r = 1 \), and hence diverge by Theorem 22.4. However, \( a_n + b_n = 0 \) for all \( n \) so the \( n \)-th partial sum of \( \sum_{n=1}^{\infty} (a_n + b_n) \) is zero for all \( n \), giving \( \sum_{n=1}^{\infty} (a_n + b_n) = 0 \) converges.

24.9. Prove that if \( \{a_n\} \) is a decreasing sequence of positive numbers and \( \sum_{n=1}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} na_n = 0 \). Deduce that \( \sum_{n=1}^{\infty} \frac{1}{n^s} \) diverges if \( 0 \leq s \leq 1 \).

**Solution.** Let \( \epsilon > 0 \). Let \( s_n = \sum_{k=1}^{n} a_k \). Since \( \{s_n\} \) converges, \( \{s_n\} \) is Cauchy so there exists some \( N \) such that \( n, m \geq N \) implies \( |s_n - s_m| < \epsilon \). In particular, for \( n \geq N \) we have \( (n-N)a_n \leq a_{n+1} + \cdots + a_n = |s_n - s_N| < \epsilon \). Hence \( \lim_{n \to \infty} (n-N)a_n = 0 \). By Theorem 22.3, we have \( \lim_{n \to \infty} Na_n = N \lim_{n \to \infty} a_n = 0 \) too. So by Theorem 12.2, we have

\[
\lim_{n \to \infty} na_n = \lim_{n \to \infty} (n-N)a_n + \lim_{n \to \infty} Na_n = 0
\]

so we’re done.

Now, note that for \( 0 \leq s \leq 1 \), we have \( \lim_{n \to \infty} \frac{1}{n^s} = \lim_{n \to \infty} n^{1-s} \neq 0 \). By the above, this means that \( \sum_{n=1}^{\infty} \frac{1}{n^s} \) cannot converge, that is, it diverges.

25.2. Let \( \{a_n\} \) satisfy the hypotheses of the alternating series test. Let \( \{s_n\} \) denote the sequence of partial sums of the series \( \sum_{n=1}^{\infty} (-1)^{n+1} a_n \). Prove that the sequence \( \{s_{2n-1}\} \) is decreasing and bounded below by 0.

**Solution.** Note \( s_{2(n+1) - 1} = s_{2n-1} - a_{2n} + a_{2n+1} \leq s_{2n-1} \) since \( a_{2n+1} \leq a_{2n} \). Hence the sequence \( \{s_{2n-1}\} \) is decreasing.

Note \( s_{2n-1} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-3} - a_{2n-2}) + a_{2n-1} \), where each term in parenthesis is bounded below by 0, and \( a_{2n-1} \) is also bounded below by 0. Hence \( \{s_{2n-1}\} \) is bounded below by 0.
25.4. Give an example of a sequence \( \{a_n\} \) of positive numbers such that \( \lim_{n \to \infty} a_n = 0 \), but the series \( \sum_{n=1}^\infty (-1)^{n+1} a_n \) diverges.

**Solution.** Let \( a_n = \begin{cases} \frac{2}{n+1} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \). Then \( \lim_{n \to \infty} a_n = 0 \) but the \((2n - 1)\)-th partial sum of \( \sum_{n=1}^\infty (-1)^{n+1} a_n \) is equal to the \(n\)-th partial sum of \( \sum_{n=1}^\infty \frac{1}{n} \). Since \( \sum_{n=1}^\infty \frac{1}{n} \) diverges by Corollary 24.3, \( \sum_{n=1}^\infty (-1)^{n+1} a_n \) also diverges.

26.4. Prove that if \( \sum_{n=1}^\infty a_n \) converges absolutely, then \( \sum_{n=1}^\infty a_n^2 \) converges.

**Solution.** By Theorem 22.3, we have \( \lim_{n \to \infty} |a_n| = 0 \). So by Theorem 13.2, \( \{a_n\} \) is bounded. Hence \( \{a_n\} \) is bounded. We apply part (i) of Theorem 26.4 (replacing \( \{b_n\} \) with \( \{a_n\} \)) to see that \( \sum_{n=1}^\infty a_n a_n = \sum_{n=1}^\infty a_n^2 \) converges absolutely. Since all terms are non-negative, \( \sum_{n=1}^\infty a_n^2 \) converges.

26.8. Let \( \{a_n\} \) be a sequence of positive numbers. Prove that if \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L \) then \( \lim_{n \to \infty} a_n^{1/n} = L \). Deduce \( \lim_{n \to \infty} \frac{n}{(n!)^{1/n}} = e \).

**Solution.** Define the sequence \( \{b_n\} \) by \( b_1 = a_1 \) and \( b_n = \frac{a_n}{a_{n-1}} \) for \( n \geq 2 \). Since \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L \), we have \( \lim_{n \to \infty} b_n = L \). Note that \( a_n = b_1 b_2 \cdots b_n \). Applying exercise 20.20 to the sequence \( \{b_n\} \), we get

\[
\lim_{n \to \infty} a_n^{1/n} = \lim_{n \to \infty} (b_1 b_2 \cdots b_n)^{1/n} = L
\]

Now, let \( a_n = \frac{n^n}{n!} \). Note that

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)^{n+1}}{(n+1)!} \frac{n^n}{n!} = \lim_{n \to \infty} \frac{(n+1)^n}{n^n} \frac{n+1}{n} = \lim_{n \to \infty} (1 + \frac{1}{n})^n = e
\]

By the conclusion above, we have

\[
\lim_{n \to \infty} \frac{n}{(n!)^{1/n}} = \lim_{n \to \infty} a_n^{1/n} = e
\]