The following is a strategy for deciding who goes first in a game which is to be played by \( N \) players. Each of the players flips a fair coin. If either (a) all coins come out the same or (b) the number of heads and tails are identical, the flips are repeated. If not, then the players in the majority (i.e. those who flipped heads if there were more heads than tails and vice versa) drop out and the remaining players repeat the process, until only one player is left, and he/she goes first.

(a) Suppose there are 3 players. What is the probability that the number of flips is greater than 2?

Let \( X \) be the random variable equal to the number of flips.

\[
P(X > 2) = 1 - P(X = 1) - P(X = 2)
\]

\[
= 1 - \frac{6}{8} - \frac{2}{8} \cdot \frac{6}{8}
\]

\[
= \frac{1}{16}.
\]

(b) Again with 3 players, what is the expected value of the number of flips?

Note that for any positive integer \( n \), we have

\[
P(X = n) = \left( \frac{2}{8} \right)^{n-1} \cdot \frac{6}{8} = \left( \frac{1}{4} \right)^{n-1} \cdot \frac{3}{4} = 3 \left( \frac{1}{4} \right)^n.
\]

Hence the expected value of the number of flips is

\[
E[X] = \sum_{n=1}^{\infty} n \cdot P(X = n)
\]

\[
= \sum_{n=1}^{\infty} n \cdot 3 \left( \frac{1}{4} \right)^n
\]

\[
= 3 \sum_{n=1}^{\infty} n \left( \frac{1}{4} \right)^n
\]

\[
= 3 \left( \frac{1/4}{(1 - 1/4)^2} \right)
\]

since \( \sum_{n=1}^{\infty} n x^n = \frac{x}{(1 - x)^2} \) for \( |x| < 1 \)

\[
= \frac{4}{3}.
\]
How do we know that \( \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} \) for \( |x| < 1 \)? We get this by differentiating the infinite geometric series equation \( \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \) with respect to \( x \), and then multiplying both sides by \( x \).

For another method of computing the infinite sum \( \sum_{n=1}^{\infty} n \left( \frac{1}{2} \right)^n \), see Example 8b on page 156 of the book, which computes the expected value of a geometric random variable.

(c) Suppose there are 4 players. What is the expected value of the number of flips?

Note that for any positive integer \( n \), we have

\[
P(X = n) = \left( \frac{8}{16} \right)^{n-1} \cdot \frac{8}{16} = \left( \frac{1}{2} \right)^n.
\]

Hence the expected value of the number of flips is

\[
E[X] = \sum_{n=1}^{\infty} n \cdot P(X = n)
\]

\[
= \sum_{n=1}^{\infty} n \left( \frac{1}{2} \right)^n
\]

\[
= \frac{1/2}{(1 - 1/2)^2}
\]

since \( \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} \) for \( |x| < 1 \)

\[
= 2.
\]

(d) Again with 4 players, what is the probability that the process ends?

The probability that the process ends is

\[
P(X = 1) + P(X = 2) + \ldots = \sum_{n=1}^{\infty} P(X = n)
\]

\[
= \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n \text{ by part (c)}
\]

\[
= \frac{1/2}{1 - 1/2}
\]

by the formula for the sum of an infinite geometric series

\[
= 1.
\]

Hence the process ends with probability 1.
2. Suppose three numbers are chosen uniformly at random from among the integers between 0 and 100 inclusive.

(a) What is the probability that the numbers add up to 100?

Let \( X_i \) denote the value of the \( i \)-th integer for \( i = 1, 2, 3 \). Then

\[
P(\text{three add to 100}) = P(X_1 + X_2 + X_3 = 100)
\]

\[
= \sum_{k=0}^{100} P(X_1 + X_2 + X_3 = 100 | X_1 = k)P(X_1 = k)
\]

\[
= \sum_{k=0}^{100} P(X_2 + X_3 = 100 - k)P(X_1 = k)
\]

\[
= \sum_{k=0}^{100} (101 - k) \left( \frac{1}{101} \right)^2 \left( \frac{1}{101} \right)
\]

\[
= \frac{1}{101^3} \sum_{k=0}^{100} (101 - k)
\]

\[
= \frac{1}{101^3} \cdot \frac{101(101 + 1)}{2}
\]

\[
= \frac{51}{101^2}
\]

\[
\approx 0.005.
\]

(b) What is the probability that two of the integers are equal, conditioned on the requirement that the three integers add up to 100?

Note that 100 is not divisible by three, it is not possible that all three integers are equal and add up to 100. The probability that two of the integers are equal and the three integers add up to 100 is

\[
P(\text{two are equal and three add to 100})
\]

\[
= 3P(X_1 = X_2, X_1 + X_2 + X_3 = 100)
\]

\[
= 3 \sum_{k=0}^{50} P(X_1 + X_2 + X_3 = 100 | X_1 = X_2 = k)P(X_1 = X_2 = k)
\]

\[
= 3 \sum_{k=0}^{50} P(X_3 = 100 - 2k)P(X_1 = X_2 = k)
\]

\[
= 3 \sum_{k=0}^{50} \left( \frac{1}{101} \right) \cdot \left( \frac{1}{101} \right)^2
\]

\[
= 3 \cdot \frac{51}{101^3}.
\]
Hence
\[ P(\text{two are equal|three add to 100}) = \frac{P(\text{two are equal and three add to 100})}{P(\text{three add to 100})} \]
\[ = \frac{3 \cdot 51/101^3}{51/101^2} \]
\[ = \frac{3}{101} \approx 0.0297 \]

(c) Given again that the sum of the integers is 100, what is the probability that all three are equal?

Since 100 is not divisible by three, it is not possible that all three integers are equal and add up to 100. So the conditional probability that all three are equal given the sum of the integers is 100 is zero.

(d) Given that the sum of all the integers is 100, what is the probability that the first number drawn is a 7?

We have
\[ P(X_1 = 7, X_1 + X_2 + X_3 = 100) = P(X_1 = 7, X_2 + X_3 = 93) \]
\[ = P(X_1 = 7)P(X_2 + X_3 = 93) \quad \text{by independence} \]
\[ = \frac{1}{101} \cdot \frac{94}{101^2} \]
\[ = \frac{94}{101^3}. \]

Thus the conditional probability that the first number drawn is a 7 given that the three integers add up to 100 is
\[ P(X_1 = 7|X_1 + X_2 + X_3 = 100) = \frac{P(X_1 = 7, X_1 + X_2 + X_3 = 100)}{P(X_1 + X_2 + X_3 = 100)} \]
\[ = \frac{94/101^3}{51/101^2} \]
\[ = \frac{94}{5151} \approx 0.0182. \]

3. Let \( S = \{1, \ldots, n\} \). Suppose that subsets \( A \) and \( B \) are chosen independently from \( S \) so that \( A \) and \( B \) are equally likely to be any of the \( 2^n \) subsets of \( S \) (including both the empty set and \( S \) itself).
(a) What is the probability that $A \subseteq B$?

There are a number of ways to approach this problem.

**Solution 1.** The easiest method is to do the following. For $1 \leq i \leq n$, let $C_i$ be the event that $A \cap \{i\} \subseteq B \cap \{i\}$. Note that $P(C_i) = \frac{3}{4}$. This is because $C_i$ occurs if $i$ is in neither $A$ nor $B$, if $i$ is in both $A$ and $B$, and if $i$ is in $B$ but not in $A$, and because $C_i$ does not occur if $i$ is in $A$ but not $B$. Note also that for $i \neq j$, the events $C_i$ and $C_j$ are independent.

Thus

$$P(A \subseteq B) = P(C_1 \text{ and } C_2 \ldots \text{ and } C_n)$$

$$= P(C_1) \cdot P(C_2) \cdots \cdot P(C_n) \text{ by independence}$$

$$= \left(\frac{3}{4}\right)^n$$

**Solution 2.** A harder approach is to condition on the size of $B$ (or $A$):

$$P(A \subseteq B) = \sum_{i=0}^{n} P(A \subseteq B | \#B = i) P(\#B = i).$$

There are $2^n$ subsets of $S$, of which $\binom{n}{i}$ have size $i$, so $P\{\#B = i\} = \binom{n}{i} / 2^n$. If $\#B = i$, then $B$ has $2^i$ subsets, so the conditional probability that $A \subseteq B$ is $2^i / 2^n$. Thus

$$P\{A \subseteq B\}$$

$$= \sum_{i=0}^{n} \frac{2^i \binom{n}{i}}{2^n 2^n}$$

$$= \frac{1}{4^n} \sum_{i=0}^{n} \binom{n}{i} 2^i$$

$$= \frac{1}{4^n} \cdot 3^n \text{ since we can evaluate } \sum_{i=0}^{n} \binom{n}{i} x^i = (1 + x)^n \text{ at } x = 2$$

$$= \left(\frac{3}{4}\right)^n.$$

(b) What is the probability that $A \cap B = \emptyset$?

Again, there are a number of ways to solve this problem.

**Solution 1.** For $1 \leq i \leq n$, let $D_i$ be the event that $A \cap B \cap \{i\} = \emptyset$. Note that $P(D_i) = \frac{3}{4}$. This is because $D_i$ occurs if $i$ is in neither $A$ nor $B$, if $i$ is
in A but not B, if i is in B but not in A, and because $D_i$ does not occur if i is in both A and B. Note also that for $i \neq j$, the events $D_i$ and $D_j$ are independent.

Thus

$$P(A \cap B = \emptyset) = P(D_1 \text{ and } D_2 \ldots \text{ and } D_n)$$

$$= P(D_1) \cdot P(D_2) \cdots \cdot P(D_n) \text{ by independence}$$

$$= \left(\frac{3}{4}\right)^n$$

**Solution 2.** Alternatively, we can observe

$$P(A \cap B = \emptyset) = P(A \subseteq S \setminus B)$$

$$= P(A \subseteq B) \text{ because } B \text{ and } S \setminus B \text{ have identical distributions}$$

$$= \left(\frac{3}{4}\right)^n \text{ by part (a).}$$

(c) What is the expected value of $\#(A \cap B)$, where $\#(A \cap B)$ is the number of elements in the set $A \cap B$?

For $1 \leq i \leq n$, let $F_i = \#(A \cap B \cap \{i\})$. Note that

$$E[F_i] = p(F_i = 1) = \frac{1}{4},$$

and that

$$\#(A \cap B) = F_1 + F_2 + \cdots + F_n.$$  

Because the expected value of a sum is the sum of the expected values, we have that

$$E[\#(A \cap B)] = E[F_1 + F_2 + \cdots + F_n]$$

$$= E[F_1] + E[F_2] + \cdots + E[F_n]$$

$$= \frac{1}{4} + \frac{1}{4} + \cdots + \frac{1}{4}$$

$$= \frac{n}{4}$$

4. Suppose that we have two tests, $T_1$ and $T_2$, for a disease. A positive test result is indicative of the presence of the disease. The following table gives the probabilities
that a healthy patient or a patient with the disease will receive a positive test result under test $T_1$ or $T_2$.

Probability of a positive test result

<table>
<thead>
<tr>
<th></th>
<th>Healthy</th>
<th>Disease</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1$</td>
<td>10%</td>
<td>95%</td>
</tr>
<tr>
<td>$T_2$</td>
<td>8%</td>
<td>90%</td>
</tr>
</tbody>
</table>

We assume the tests are independent, so that the events of $T_1$ and $T_2$ being positive are independent. Assume also that 5% of the population has the disease.

(a) Find the conditional probabilities $P(\text{Healthy}|T_i \text{ positive})$ for $i = 1, 2$.

Let $H = \text{Healthy}$ be the event that the patient is healthy, and let $D = \text{Disease}$ be the event that the patient has the disease.

\[
P(H|T_1 \text{ positive}) = \frac{P(H \text{ and } T_1 \text{ positive})}{P(T_1 \text{ positive})} = \frac{P(T_1 \text{ positive}|H)P(H)}{P(T_1 \text{ positive}|H)P(H) + P(T_1 \text{ positive}|D)P(D)} = \frac{0.10 \cdot 0.95}{0.10 \cdot 0.95 + 0.95 \cdot 0.05} = \frac{2}{3}
\]

Similarly,

\[
P(H|T_2 \text{ positive}) = \frac{P(T_2 \text{ positive}|H)P(H)}{P(T_2 \text{ positive}|H)P(H) + P(T_2 \text{ positive}|D)P(D)} = \frac{0.08 \cdot 0.95}{0.08 \cdot 0.95 + 0.90 \cdot 0.05} \approx 0.6281
\]
(b) Find the conditional probability \( P(\text{Healthy}|T_1 \text{ and } T_2 \text{ are both negative}) \).

\[
P(H|T_1, T_2 \text{ negative}) \\
= \frac{P(H \text{ and } T_1, T_2 \text{ negative})}{P(T_1, T_2 \text{ negative})} \\
= \frac{P(T_1, T_2 \text{ negative}|H)P(H)}{P(T_1 \text{ negative}|H)P(T_2 \text{ negative}|H)P(H) + P(T_1 \text{ negative}|D)P(T_2 \text{ negative}|D)P(D)} \\
= \frac{P(T_1 \text{ negative}|H)P(T_2 \text{ negative}|H)P(H)}{P(T_1 \text{ negative}|H)P(T_2 \text{ negative}|H)P(H) + P(T_1 \text{ negative}|D)P(T_2 \text{ negative}|D)P(D)} \\
\text{by independence} \\
= \frac{0.90 \cdot 0.92 \cdot 0.95}{0.90 \cdot 0.92 \cdot 0.95 + 0.05 \cdot 0.10 \cdot 0.05} \\
\approx 0.99968
\]

(c) Find the conditional probability \( P(\text{Disease}|T_1 \text{ and } T_2 \text{ are both positive}) \).

\[
P(D|T_1, T_2 \text{ positive}) \\
= \frac{P(D \text{ and } T_1, T_2 \text{ positive})}{P(T_1, T_2 \text{ positive})} \\
= \frac{P(T_1 \text{ positive}|D)P(D)}{P(T_1 \text{ positive}|D)P(D) + P(T_1 \text{ positive}|H)P(H)} \\
= \frac{P(T_1 \text{ positive}|D)P(T_2 \text{ positive}|D)P(D) + P(T_1 \text{ positive}|H)P(T_2 \text{ positive}|H)P(H)}{P(T_1 \text{ positive}|D)P(T_2 \text{ positive}|D)P(D) + P(T_1 \text{ positive}|H)P(T_2 \text{ positive}|H)P(H)} \\
\text{by independence} \\
= \frac{0.95 \cdot 0.90 \cdot 0.05}{0.95 \cdot 0.90 \cdot 0.05 + 0.10 \cdot 0.08 \cdot 0.95} \\
\approx 0.8491
\]

5. Let \( X \) be a binomial random variable with number of trials equal to \( n \) and probability of success \( p \). What value of \( p \) maximizes \( P\{X = k\} \), for \( k = 0, 1, \ldots, n \)?

Note that

\[
P\{X = k\} = \binom{n}{k} p^k (1 - p)^{n-k}.
\]

Our goal is to maximize this as a function of \( p \in [0, 1] \). First let’s consider the boundary points \( p = 0 \) or \( 1 \). It’s clear that \( p = 0 \) is the optimum choice for \( k = 0 \), that \( p = 1 \) is the optimum choice for \( k = n \), and that \( p = 0 \) or \( 1 \) is not the optimum choice for any other value of \( k \).
Next, we take the derivative of this function and set it equal to zero, to get

\[ 0 = \frac{d}{dp} \left( P\{X = k\} \right) \]
\[ = \frac{d}{dp} \left( \binom{n}{k} p^k (1 - p)^{n-k} \right) \]
\[ = \binom{n}{k} \left( kp^{k-1}(1 - p)^{n-k} - (n - k)p^k(1 - p)^{n-k-1} \right) \]
\[ = \binom{n}{k} p^{k-1}(1 - p)^{n-k-1} \left( k(1 - p) - (n - k)p \right). \]

We already considered the boundary cases \( p = 0 \) or \( 1 \), so we can assume \( p \neq 0, 1 \). Hence for the derivative to equal zero, we need

\[ 0 = k(1 - p) - (n - k)p \]
\[ \implies 0 = k - kp - np + kp \]
\[ \implies 0 = k - np \]
\[ \implies p = \frac{k}{n} \]

Hence the value \( p = \frac{k}{n} \) maximizes \( P\{X = k\} \), for \( k = 0, 1, \ldots, n \).

6. We suppose that a point \((x, y)\) is chosen from the circular region \( D = \{(x, y) \mid x^2 + y^2 \leq 1\} \), in such a way that the probability that the point is chosen from any region in \( D \) is proportional to its area.

(a) What is the probability that \( y \geq \frac{1}{2} \)?

Note the area of the circular region \( D \) is \( \pi \).

The region \( y \geq \frac{1}{2} \) is drawn in blue below.
To get the area of the blue region, we take the area of the blue and green regions and subtract the area of the green region. The blue and green regions form a sector of the circle with angle $60^\circ + 60^\circ = 120^\circ$, and hence have combined area $\frac{120}{360} \pi = \frac{\pi}{3}$. The green region is a triangle with area $\frac{1}{2} \cdot \sqrt{3} \cdot \frac{1}{2} = \frac{\sqrt{3}}{4}$. Hence the area of the blue region is $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$.

The probability that $y \geq \frac{1}{2}$ is equal to the area of the blue region divided by the area of $D$, or

$$\frac{\frac{\pi}{3} - \frac{\sqrt{3}}{4}}{\pi} = \frac{1}{3} - \frac{\sqrt{3}}{4\pi} \approx 0.1955$$

(b) What is the conditional probability that $y \leq 0$ given that $x \geq \frac{1}{2}$?

$$P(y \leq 0|x \geq \frac{1}{2}) = \frac{P(y \leq 0 \text{ and } x \geq \frac{1}{2})}{P(x \geq \frac{1}{2})}$$

= area of blue region

= area of blue region and red regions

= $\frac{1}{2}$ by symmetry.

(c) What is the probability that $(x, y)$ lies inside the square $S$ given by $-\frac{1}{\sqrt{2}} \leq x, y \leq \frac{1}{\sqrt{2}}$?
(d) What is the conditional probability that \( x + y \geq 1 \) given that \((x,y)\) is in \( S \)?

\[
P(x + y \geq 1|(x,y) \text{ in } S) = \frac{\text{area of blue region}}{\text{area of red and blue regions}}
\]
\[
= \frac{\frac{1}{2} \cdot (\sqrt{2} - 1) \cdot (\sqrt{2} - 1)}{\sqrt{2} \cdot \sqrt{2}}
\]
\[
= \frac{3 - 2\sqrt{2}}{4}
\]
\[
\approx 0.429.
\]

7. Let \( p(n) = \frac{2}{3^n} \) for any integer \( n \geq 1 \). Let \( X \) denote the random variable \( X(n) = n \).
(a) Show that \( p \) defines a probability distribution on the set \( \{1, 2, \ldots, n, \ldots\} \).

Note that \( p(n) = \frac{2}{3^n} \geq 0 \) for all \( n \geq 1 \), as required. Note also that

\[
\sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{2/3}{1 - 1/3} = 1.
\]

Hence \( p \) defines a probability distribution on the set \( \{1, 2, \ldots, n, \ldots\} \).

(b) Determine the expected value of \( X \).

\[
E[X] = \sum_{n=1}^{\infty} n \cdot \frac{2}{3^n}
= 2 \sum_{n=1}^{\infty} n \left( \frac{1}{3} \right)^n
= 2 \left( \frac{1/3}{(1 - 1/3)^2} \right) \quad \text{since } \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} \text{ for } |x| < 1
= \frac{3}{2}.
\]

How do we know that \( \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} \) for \( |x| < 1 \)? We get this by differentiating the infinite geometric series equation \( \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \) with respect to \( x \), and then multiplying both sides by \( x \).

For another method of computing the infinite sum \( \sum_{n=1}^{\infty} n \left( \frac{1}{3} \right)^n \), see Example 8b on page 156 of the book, which computes the expected value of a geometric random variable.

(c) Determine the variance of \( X \).

Before we begin, let’s derive a formula we will need later. As described in part (a), for \( |x| < 1 \) we know that

\[
\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.
\]

If we differentiate both sides with respect to \( x \), we get

\[
\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} = \frac{1 - x^2}{(1-x)^4}.
\]

Multiplying both sides by \( x \), we get

\[
\sum_{n=1}^{\infty} n^2 x^n = \frac{x - x^3}{(1-x)^4}.
\]
Now,

\[
\text{Var}(X) = E[X^2] - (E[X])^2
\]

\[
= \left( \sum_{n=1}^{\infty} n^2 \cdot \frac{2}{3^n} \right) - \left( \frac{3}{2} \right)^2
\]

\[
= 2 \left( \sum_{n=1}^{\infty} n^2 \left( \frac{1}{3} \right)^n \right) - \frac{9}{4}
\]

\[
= 2 \left( \frac{1/3 - (1/3)^3}{(1 - 1/3)^4} \right) - \frac{9}{4}
\]

since \( \sum_{n=1}^{\infty} n^2 x^n = \frac{x - x^3}{(1 - x)^4} \) for \( |x| < 1 \)

\[
= 3 - \frac{9}{4}
\]

\[
= \frac{3}{4}.
\]

(d) What is the conditional probability that \( n \) is divisible by 4 given that it is divisible by 2?

\[
P(4 \text{ divides } n | 2 \text{ divides } n) = \frac{P(4 \text{ divides } n \text{ and } 2 \text{ divides } n)}{P(2 \text{ divides } n)}
\]

\[
= \frac{P(4 \text{ divides } n)}{P(2 \text{ divides } n)}
\]

\[
= \frac{\sum_{k=1}^{\infty} \frac{2}{3^{4k}}}{\sum_{k=1}^{\infty} \frac{2}{3^{2k}}}
\]

\[
= \frac{2/81}{1 - 1/9} - \frac{2/9}{1 - 1/9}
\]

\[
= \frac{1}{40}
\]

\[
= \frac{1}{10}.
\]

8. Suppose that 10,000 monkeys are typing at typewriters, and that 4,000 of them are typing at 100 characters per minute and 6,000 of them are typing at 200 characters per minute. After how many years does it become likely that one of the monkeys will have typed “Macbeth”?

 Comments:

• Assume that “Macbeth” is a document with 300,000 alphabetic characters. A monkey must type “Macbeth” from start to finish. For a monkey to success-
fully type “Macbeth”, there can be no typos within the 300,000 characters of “Macbeth”, but the text before or after “Macbeth” can be gibberish.

- Your answer should only be an estimate, and will require assumptions about the document, so different estimates will be possible. State clearly all assumptions you make.

First let’s state our assumptions. We assume that there are 26 characters in Macbeth (we are ignoring punctuation and spaces) and that there are 26 characters on each keyboard. We assume that a monkey types a key by choosing one of the 26 characters uniformly at random. We assume that the monkeys are able to continue typing at their paces indefinitely. We assume that the monkeys are typing independently of each other.

Moreover, we assume that whenever we compare a copy of “Macbeth” with a copy shifted by a fixed amount, the two copies never coincide on the region of overlap. This allows us to assume that the event that “a monkey types Macbeth starting at his $i$-th character” is weakly independent of the event that “the same monkey types Macbeth starting at his $j$-th character”, even when $i$ and $j$ are such that the two copies of Macbeth would overlap. Hence we can treat each of the events

$$E_i(j) = \text{“Monkey } j \text{ types Macbeth beginning at his } i\text{-th character”},$$

indexed by $i$ and $j$, as independent of each other.

Next, let’s compute how many trials we obtain in a year. There are 525,600 minutes in a year. The monkeys who type 100 characters per minute will type $5.256 \times 10^7$ characters in a year. The monkeys who type 200 characters per minute will type $1.0512 \times 10^8$ characters in a year. This means that the first and second kinds of monkey will perform approximately $5.256 \times 10^7$ and $1.0512 \times 10^8$ trials per year, respectively. Since there are 4,000 monkeys of the first type and 6,000 of the second, together they will perform approximately

$$4,000 \cdot 5.256 \times 10^7 + 6,000 \cdot 1.0512 \times 10^8 \approx 8.5 \times 10^{11}$$

trials during a year.

Lastly, we need to see how many trials are required before the probability that one of the monkeys will type Macbeth is at least $\frac{1}{2}$. We will use the Poisson approximation.
The probability of success for any given trial is \( p = \frac{1}{26^{300,000}} \). Let \( n \) be the number of trials. We need to choose \( n \) such that

\[
\frac{1}{2} = P(\text{at least one success}) \approx \exp\left(-n \cdot \frac{1}{26^{300,000}}\right).
\]

Taking the natural log of both sides, we see that we need

\[
\frac{-n}{26^{300,000}} \approx \ln\left(\frac{1}{2}\right)
\]

or

\[
\frac{n}{26^{300,000}} \approx -\ln\left(\frac{1}{2}\right) = \ln(2)
\]

or

\[ n \geq \ln(2) \cdot 26^{300,000} \]

Recall that \( n \) is the number of trials we need so that the probability that one of the monkeys will type Macbeth is at least \( \frac{1}{2} \). Since there are approximately \( 8.5 \times 10^{11} \) trials per year, our estimate is that the number of years until it becomes likely that one of the monkeys will type Macbeth is

\[
\frac{n}{8.5 \times 10^{11}} = \frac{\ln(2) \cdot 26^{300,000}}{8.5 \times 10^{11}}
\]

years.