ColoState Spring 2015 Math 340 Exam 1

Thu. 03/05/2015

Name: Answers CSUID: 100/100

Section: _____

Problem	Score
1	10
2	15
3	15
4	15
5	15
6	15
7	15
Total	100

Exam Policy

(i) **No** calculator, textbook, homework, or any other references should be used. Please write down all necessary steps, partial credit will be given if deserved.

(ii) You could use one letter-size 2-sided Cheat Sheet for this exam.

Good luck!

(10 points) *Problem 1.* True or False, circle your answer (2 points for each item, no partial credit).

(i) (T) (F) The function $x(t) = \tan(t) + C$ is a solution of the ODE $x'(t) = 1 + x^2$ for any constant C.

(ii) (T) (F) The linear ODE $tx'(t) = x + 3t^2$ has a unique solution satisfying the initial condition $x(0) = 1$.

(iii) (T) (F) The function $\mu(x) = x$ is an integrating factor for the non-exact ODE $(xy - 1)dx + (x^2 - xy)dy = 0$.

(iv) (T) (F) Let $\mathbf{u} = (2, 2, 1), \mathbf{v} = (1, 1, 2), \mathbf{w} = (4, 4, 2)$ be three vectors. Then the subspace spanned by $\mathbf{u}, \mathbf{v}, \mathbf{w}$ has dimension 3.

(v) (T) (F) Let $\mathbf{u} = (2, 2, 1), \mathbf{v} = (1, 1, 2), \mathbf{w} = (4, 4, 2)$ be three vectors. Then $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent.

(15 points) *Problem 2.* A container of fluid is placed in a refrigerator and allowed to cool. Assume the temperature in the refrigerator is 40 degrees F (ahrenheit) and the initial temperature of the fluid is 180 degrees F.

$A = 40°F$
$T(0) = T_0 = 180°F$

(5) (i) Write down an initial value problem whose solution describes how the temperature $T(t)$ of the fluid varies with time.

(10) (ii) If it is observed that after 10 minutes, the temperature of the fluid is 110 degrees F, then what will the temperature of the fluid be after 30 minutes?

$T(10) = 110°F$

(i) $\boxed{\dfrac{dT}{dt} = -k(T-40) \quad (+3), \qquad T(0) = 180 \quad (+2)}$

(ii) $\displaystyle\int \frac{dT}{T-40} = \int -k\,dt$

$\Rightarrow \ln|T-40| = -kt + c$

$\Rightarrow T - 40 = \hat{c}e^{-kt}$

$\Rightarrow T(t) = \hat{c}e^{-kt} + 40$ (+3)

$\Rightarrow 180 = T(0) = \hat{c} + 40$

$\Rightarrow \hat{c} = 180 - 40 = 140$

$\Rightarrow T(t) = 140e^{-kt} + 40$ (+2)

$\Rightarrow 110 = T(10) = 140e^{-k\cdot 10} + 40$

$\Rightarrow 70 = 140e^{-10k}$

$\Rightarrow \dfrac{70}{140} = e^{-10k}$

$\Rightarrow \dfrac{1}{2} = e^{-10k}$

$\Rightarrow \ln(\tfrac{1}{2}) = -10k$

$\Rightarrow \dfrac{-\ln(2)}{-10} = k$

$\Rightarrow k = \dfrac{\ln 2}{10}$ (+2)

$\Rightarrow T(t) = 140e^{-\frac{\ln 2}{10}t} + 40$ (+1)

$\Rightarrow T(30) = 140e^{-\frac{\ln 2}{10}\cdot 30} + 40$

$= 140e^{-3\ln 2} + 40$

$= 140e^{\ln 2^{-3}} + 40$

$= 140(2^{-3}) + 40$

$= \dfrac{140}{2^3} + 40$

(+1)

$= \dfrac{140}{8} + 40$

$= 17.5 + 40$

$= \boxed{57.5°F}$ (+1)

$\frac{115}{2} = 57$

$8\overline{)140}$ = 17.5
-8
60
56
40
40
0

(15 points) *Problem 3.* Consider a linear ODE $(1+t^2)x'(t) + 4tx(t) = (1+t^2)^{-2}$.

10 (i) Find the general solution.

5 (ii) Find the particular solution satisfying $x(1) = 0$.

Normal form: $x'(t) + \frac{4t}{1+t^2}x = (1+t^2)^{-3}$.

I.F: $u(t) = e^{\int \frac{4t}{1+t^2}dt} = e^{\int \frac{2\,du}{u}}$ $\quad 1+t^2 = u$, $2t\,dt = du$

$= e^{2\ln u}$

3 → $= (1+t^2)^2$.

2 → $\therefore \frac{d}{dt}\left[(1+t^2)^2 x(t)\right] = (1+t^2)^{-1}$.

Integrating $(1+t^2)^2 x(t) = \int (1+t^2)^{-1}dt$

3 → $= \int \frac{1}{1+t^2}dt = \tan^{-1}t + C$

5 2 → $\boxed{x(t) = \frac{\tan^{-1}t}{(1+t^2)^2} + \frac{C}{(1+t^2)^2}}$

2 → $x(1) = 0 = \frac{\tan^{-1}1}{(1+1)^2} + \frac{C}{(1+1)^2} \Rightarrow C = \frac{\pi}{4}$ ← 2

1 → $\therefore x(t) = \frac{\tan^{-1}t + \pi/4}{(1+t^2)^2}$.

P $Q=-(2x^2+y)$

(15 points) *Problem 4.* It is known that the ODE $2xydx-(2x^2+y)dy=0$ is not exact.

8pts (i) Find an integrating factor in the form $\mu(y)$ for the ODE.

5pts (ii) Use the integrating factor to find a general solution for the ODE.

2pts (iii) Sketch the solution curve that passes the point $(0,-1)$.

(i) $\mu(y)=e^{-\int g(y)dy}$ for $g(y)=\frac{1}{P}(P_y-Q_x)=\frac{1}{2xy}(2x-(-4x))=\frac{3}{y}$ 3pts.

$\mu(y)=e^{-\int\frac{3}{y}dy}=e^{-3\ln(y)}=\frac{1}{y^3}$ 2pts, 3pts

NOTE: If wrong μ at most
(i) 4/8 pts
(ii) 2/5 pts
(iii) 0/2 pts

Check:

$$\frac{2x}{y^2}dx+\left(-\frac{2x^2}{y^3}-\frac{1}{y^2}\right)dy=0$$

$P_y=-\frac{4x}{y^3}$ $Q_x=-\frac{4x}{y^3}$ $P_y=Q_x$ ✓

(ii) $F(x,y)=\int\frac{2x}{y^2}dx=\frac{x^2}{y^2}+\varphi(y)$ 2pts.

$\frac{\partial F}{\partial y}=-\frac{2x^2}{y^3}+\varphi'(y)=-\frac{2x^2}{y^3}-\frac{1}{y^2}$ 1pt. $\Rightarrow \varphi'(y)=-\frac{1}{y^2}$

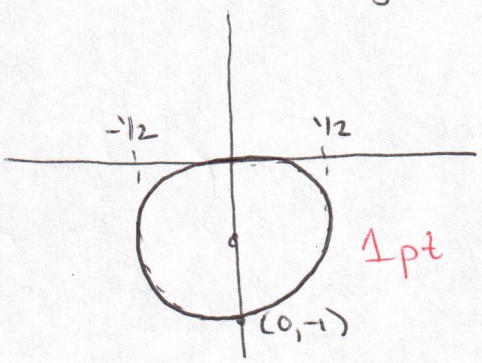
$\Rightarrow \varphi(y)=\frac{1}{y}$ 1pt

$F(x,y)=\frac{x^2}{y^2}+\frac{1}{y}=C$ 1pt

(iii) $F(0,-1)=\frac{0}{1}-\frac{1}{1}=C \Rightarrow C=-1$ 1pt

Then $\frac{x^2}{y^2}+\frac{1}{y}=-1 \Rightarrow x^2+y+y^2=0 \Rightarrow x^2+(y+\frac{1}{2})^2=\frac{1}{4}$

circle, center at $(0,-1/2)$, $R=\frac{1}{2}$



1pt

way 2

$F(x,y)=\int\left(-\frac{2x^2}{y^3}-\frac{1}{y^2}\right)dy=\frac{x^2}{y^2}+\frac{1}{y}+\varphi(x)$

$\frac{\partial F}{\partial x}=\frac{2x}{y^2}+\varphi'(x)=\frac{2x}{y^2} \Rightarrow \varphi'(x)=0$

$F(x,y)=\frac{x^2}{y^2}+\frac{1}{y}=C$

(15 points) *Problem 5.* Determine whether the following statements are true (T) or false (F) (3 points each item, no partial credit).

Consider the ODE $\sqrt{t}\frac{dx}{dt} = \sqrt{x}$.

(i) (T) (F) The ODE has a unique solution satisfying the initial condition $x(0) = 0$.

(ii) (T) (F) The ODE has a unique solution satisfying the initial condition $x(0) = 1$.

(iii) (T) (F) The ODE has a unique solution satisfying the initial condition $x(1) = 1$.

(iv) (T) (F) The ODE has a unique solution satisfying the initial condition $x(1) = 0$.

(v) (T) (F) The ODE has a unique solution satisfying the initial condition $x(a) = b$ as long as we have $a > 0$ and $b > 0$.

(i) $x(t) \equiv 0$ (const. fxn.) $\rightarrow$ both satisfy the initial condition and the ODE.

$x(t) = t$

(ii) Actually no solution

(iii) the uniqueness Thm. applies

(iv) Actually there are two solutions

$x(t) \equiv 0$ (const. fxn.)

$x(t) = (\sqrt{t}-1)^2$ (obtained from separation of variables)

(v) the uniqueness Thm. applies

(15 points) *Problem 6.* Consider the ODE $x'(t) = (1-x)\ln(1+x)$.

(i) Find all equilibrium solutions and determine the stability of each.

(ii) Sketch two solution curves on the (t, x)-plane that are not straight lines.

(iii) Let $x_p(t)$ be the solution satisfying $x_p(0) = \frac{1}{2}$, find $\lim_{t\to+\infty} x_p(t)$. Briefly justify your answer.

7

(i) $f(x) = (1-x)\ln(1+x) \overset{\text{set}}{=} 0$

$x = 1, 0$ (+1 each)

rest of points for clear work

$f'(x) = (1-x)\left(\frac{1}{1+x}\right) + \ln(1+x)(-1)$

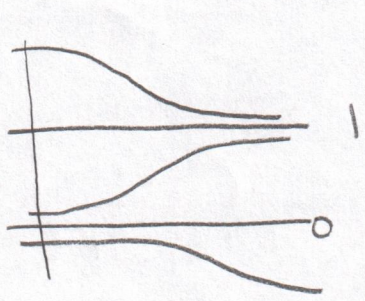
$f'(1) = -\ln(2)$ stable

$f'(0) = 1$ unstable

(+1 each)

4

(ii)



(+2 each curve)

half-credit if incorrect but ok based on (i)

4

(iii) $\lim_{t\to\infty} x_p(t) = 1$ since $x = 1$ is a stable solution and $x = 0$ is an unstable solution

(+2 limit)

(+2 explanation)

half-credit if incorrect but ok based on (ii)

(15 points) *Problem 7.* Consider the linear system $\mathbf{Ax} = \mathbf{b}$ as follows

$$\begin{cases} 4x_1 + 2x_2 - 4x_3 = 8 \\ 2x_1 + x_2 - 2x_3 = 4 \\ 4x_1 + 2x_2 - 4x_3 = 8 \end{cases}$$

(i) Apply elementary row operations to simplify the augmented matrix to the reduced row echelon form.

(ii) Write down the solutions of the linear system in a parametric form.

(iii) Find a basis for null($\mathbf{A}$).

(i) (1 pt)

$$\left[\begin{array}{ccc|c} 4 & 2 & -4 & 8 \\ 2 & 1 & -2 & 4 \\ 4 & 2 & -4 & 8 \end{array}\right] \xrightarrow{\substack{R3-R1\to R3 \\ R1+-2R2\to R1}} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R1\leftrightarrow R2}$$

/5pts

(3pt)

$$\left[\begin{array}{ccc|c} 2 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R1\to R1} \left[\begin{array}{ccc|c} 1 & 1/2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ (1 pt)

(ii) 2 free variables:

$x_2 = s$ (1)

$x_3 = t$

/5 pts

$2x_1 + s - 2t = 4$

$x_1 = 2 - \frac{1}{2}s + t$ (1)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(1) (1) (1)

(iii) $\left\{\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$

/5pts

-3pts for error carried forward from (ii)

small sign error carried through -1