Fourier Series Methods, Part II

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1 Even and Odd Functions

Recall that if $f(-t) = f(t)$, then we say $f(t)$ is even. Conversely, if $f(-t) = -f(t)$, then we say $f(t)$ is odd.

Example 1.0.1. *$f(t) = t^2$ and $g(t) = \cos(t)$ are even functions.*
$f(t) = t^3$ and $g(t) = \sin(t)$ are odd functions.
$f(t) = t^2 + t^3$ is neither even, nor odd.

Proposition 1.1. *Suppose $f(t)$ is a piecewise continuous periodic function with period $2L$. Let*

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\pi t/L) + b_n \sin(n\pi t/L),$$

be the Fourier series for $f(t)$.

- *If $f(t)$ is even, then $b_n = 0$ for all n.*
- *If $f(t)$ is odd, then $a_n = 0$ for all n.*

This is fairly easy to prove using the fact that the sine and cosine functions are odd and even respectively, and taking into consideration the product of even and odd functions along with their integrals from $-L$ to L.

Our goal is to use the Fourier series representation of a function to solve a differential equation. Frequently, this differential equation will be defined for $0 < t < L$. To compute the Fourier transform using the formulas we have, we need a function defined for $-L < t < L$. We have to make a choice on $-L < t < 0$. It is often easiest to take the even or odd extension of $f(t)$.

Example 1.0.2. *Consider $f(t) = t$ for $0 < t < L$. The even extension of $f(t)$ is given by*

$$f_{even}(t) = \begin{cases} -t & -L < t < 0 \\ t & 0 < t < L \end{cases}$$

The odd extension of $f(t)$ is given by

$$f_{odd}(t) = \begin{cases} t & -L < t < 0 \\ t & 0 < t < L \end{cases}$$

Making the choice of an even or odd extension leads to an interesting consequence in the Fourier series. Namely, that the even extension gives an even function causing $b_n = 0$ for all n in the Fourier series. Conversely, the odd extension gives an odd function causing $a_n = 0$ for all n. This gives rise to the following definition.

Definition 1.1. *Suppose that $f(t)$ is piecewise continuous on $0 < t < L$. The Fourier cosine series of f is*

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi t/L),$$

with

$$a_n = \frac{2}{L} \int_{-L}^{L} f(t) \cos(n\pi t/L) dt.$$

The Fourier sine series of f is

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t/L),$$

with

$$b_n = \frac{2}{L} \int_{-L}^{L} f(t) \sin(n\pi t/L) dt.$$

Although both of these converge to $f(t)$ it is sometimes the case that one choice is preferable over the other. Suppose you know $f(0) = 1$ but you try to use the Fourier sine series. The series will converge to f everywhere except at $t = 0$, where it will always be zero. This is not ideal if $f(0) = 1$ is an initial condition for a differential equation. We will discuss this more later.

2 Sequences of Functions and Convergence of Derivatives

We know how to find the Fourier series for a particular function, but we have not discussed how this can be used to solve a differential equation. There is a good reason for this. To illustrate the issue, consider the following question: if $f_N(t)$ converges to $f(t)$ as $N \to \infty$, does $f'_N(t)$ converge to $f'(t)$ as $N \to \infty$? The answer may surprise you...as this is not always true.

Example 2.0.3. *Consider*

$$f_N(t) = \frac{\sin(nx)}{\sqrt{n}}.$$

One can use the Squeeze Theorem to see that $f_N(t) \to 0$ as $N \to \infty$. So the sequence converges to $f(t) = 0$. Clearly $f'(t) = 0$. However,

$$f'_N(t) = \sqrt{n}\cos(nt),$$

which does not converge at all as $N \to \infty$!

We can make certain restriction on the function $f(t)$ to exclude the functions where this occurs. In our context this is done in the following theorem.

Theorem 2.1. *Suppose that the function f is continuous for all t, periodic with period $2L$, and that f' is piecewise smooth for all t. Then the Fourier series for f' is found by differentiating the Fourier series for f.*

Fortunately, in the setting of differential equations we typically assume that the solution satisfies this criteria.

3 Fourier Series Solutions to Boundary Value Problems

We are interested in solving problems of the form

$$ax'' + bx' + cx = f(t), \quad 0 < t < L,$$

$$x(0) = x(L) = 0.$$

It may be possible to find a particular solution and a homogeneous solution as we did previously. However, it is frequently easier to use a Fourier series.

First we will write $f(t)$ in terms of its Fourier series

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos(n\pi t/L) + B_n \sin(n\pi t/L)).$$

Then we will assume that we can write the solution $x(t)$ in terms of its Fourier series

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\pi t/L) + b_n \sin(n\pi t/L)).$$

Note that A_n and B_n can be determined before considering the differential equation. Then you will use the differential equation to find a_n and b_n.

Example 3.0.4. *Find the Fourier series solution to*

$$x'' + 4x = 4t, \quad x(0) = x(1) = 0.$$

Here we have $f(t) = 4t$. We should choose a periodic extension of $f(t)$ so that the boundary conditions are satisfied. This is mainly because we will want to represent the solution $x(t)$ in terms of the same basis, and the solution must satisfy the boundary conditions. For this reason we will choose the Fourier sine series of $f(t)$, which is found to be

$$f(t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi t).$$

Therefore, we expect a solution of the form

$$x(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t).$$

If we substitute this into the differential equation we get

$$\sum_{n=1}^{\infty}(-n^2\pi^2 + 4)b_n \sin(n\pi t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi t).$$

The coefficients of each sine mode must match up, so we have

$$b_n = \frac{8(-1)^{n+1}}{n\pi(4 - n^2\pi^2)}.$$

Thus, our solution is

$$x(t) = \sum_{n=1}^{\infty}\frac{8(-1)^{n+1}}{n\pi(4 - n^2\pi^2)}\sin(n\pi t).$$

Example 3.0.5. *Find the Fourier series solution to*

$$x'' + 2x = t, \quad x'(0) = x'(\pi) = 0.$$

Here we have $f(t) = t$. The boundary conditions, $x'(0) = x'(\pi) = 0$ suggest that the Fourier cosine series is appropriate here. The Fourier cosine series of $f(t)$ is found to be

$$f(t) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{\pi n^2}\cos(nt).$$

We then assume that $x(t)$ has the form

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt).$$

Substituting this into the differential equation gives,

$$\sum_{n=1}^{\infty}(-n^2 + 2)a_n \cos(nt) + \frac{a_0}{2} = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{\pi n^2}\cos(nt).$$

Therefore we have,

$$a_0 = \pi, \quad a_n = \frac{(-n^2 + 2)2(-1)^n}{\pi n^2},$$

and our solution is

$$x(t) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{(-n^2 + 2)2(-1)^n}{\pi n^2}\cos(nt).$$