

M340

Exam Two

Section 2

10/13/04

Name: _____

Problem	Points
1	
2	
3	
4	
5	
Total	

1. Use the definition of the Laplace transform to find the Laplace transform of the following functions:

(a) $f(t) = e^{3t}$
(10 points)

$$\begin{aligned} L[f(t)] &= \lim_{T \rightarrow \infty} \int_0^T e^{3t} e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \int_0^T e^{-(s-3)t} dt \\ &= \lim_{T \rightarrow \infty} \left(-\frac{e^{-(s-3)t}}{s-3} \right) \Big|_0^T \\ &= \lim_{T \rightarrow \infty} \left(-\frac{e^{-(s-3)T}}{s-3} - \frac{1}{s-3} \right) \\ &= \frac{1}{s-3} \end{aligned}$$

(b) $f(t) = H(t-1) - H(t-3)$
(10 points)

$$\begin{aligned} L[f(t)] &= \int_1^3 e^{-st} dt \\ &= \left(\frac{e^{-st}}{-s} \right) \Big|_1^3 \\ &= \frac{e^{-s} - e^{-3s}}{s} \end{aligned}$$

(c) $f(t) = \delta_5(t)$
(5 points)

$$\begin{aligned} L[f(t)] &= \int_0^{\infty} \delta_5(t) e^{-st} dt \\ &= e^{-5s} \end{aligned}$$

2. When we use variation of parameters to solve a second order, linear differential equation, the formulas involve dividing by the quantity $y_1 y_2' - y_1' y_2$. Briefly explain why this quantity is never zero (in this context). (10 points)

The functions $y_1(t)$ and $y_2(t)$ are linearly independent solutions to the homogeneous equation. The quantity $y_1 y_2' - y_1' y_2$ is the Wronskian of y_1 and y_2 which is never zero when the functions are linearly independent.

3. Use the method of undetermined coefficients to find the general solution to the following differential equation. (20 points)

$$y'' + 6y' + 9y = 3t + 4$$

First we find the homogeneous solution.

$$y'' + 6y' + 9y = 0$$

$$\lambda^2 + 6\lambda + 9 = 0$$

$$(\lambda + 3)^2 = 0$$

This has repeated real roots, so the general form of the solution to the homogeneous equation is

$$y_h(t) = C_1e^{-3t} + C_2te^{-3t}.$$

Next we guess the form of the particular solution to be

$$y_p(t) = at + b,$$

based on the form of the forcing function. Substituting this into the differential equation gives:

$$(0) + 6(a) + 9(at + b) = 3t + 4.$$

This leads to two equations and two unknowns:

$$9a = 3t, \quad 6a + 9b = 4.$$

We find $a = \frac{1}{3}$ and $b = \frac{2}{9}$. This makes the particular solution

$$y_p(t) = \frac{1}{3}t + \frac{2}{9},$$

and the general solution

$$y(t) = \frac{1}{3}t + \frac{2}{9} + C_1e^{-3t} + C_2te^{-3t}.$$

4. Use variation of parameters to find the general solution to the following differential equation. (20 points)

$$y'' + 3y' + 2y = 6e^{4t}$$

First we find the homogeneous solution.

$$y'' + 3y' + 2y = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 1)(\lambda + 2) = 0$$

This has distinct real roots, so the general form of the solution to the homogeneous equation is

$$y_h(t) = C_1 e^{-t} + C_2 e^{-2t}.$$

We now assume the particular solution has the form

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t),$$

where

$$v_1 = \int \frac{-y_2 f}{y_1 y_2' - y_1' y_2} dt, \quad v_2 = \int \frac{y_1 f}{y_1 y_2' - y_1' y_2} dt.$$

First we find

$$\begin{aligned} y_1 y_2' - y_1' y_2 &= (e^{-t})(-2e^{-2t}) - (-e^{-t})(e^{-2t}) \\ &= -e^{-3t} \end{aligned}$$

Then we have

$$\begin{aligned} v_1 &= \int \frac{-e^{-2t} 6e^{4t}}{-e^{-3t}} dt \\ &= \int 6e^{5t} dt \\ &= \frac{6}{5} e^{5t} \end{aligned}$$

and,

$$\begin{aligned} v_2 &= \int \frac{e^{-t} 6e^{4t}}{-e^{-3t}} dt \\ &= \int 6e^{6t} dt \\ &= e^{6t} \end{aligned}$$

So the particular solution is

$$\begin{aligned}y_p &= \frac{6}{5}e^{5t}e^{-t} + e^{6t}e^{-2t} \\ &= \frac{1}{5}e^{4t}\end{aligned}$$

and the general solution is

$$y(t) = \frac{1}{5}e^{4t} + C_1e^{-t} + C_2e^{-2t}.$$

5. Use the Laplace transform to find the solution to the following initial value problem. (25 points)

$$y'' - 9y = f(t), \quad y(0) = 1, \quad y'(0) = -1,$$

where

$$f(t) = \begin{cases} 0 & t < 2 \\ 1 & 2 \leq t < 4 \\ 0 & 4 \leq t \end{cases}$$

Taking the Laplace transform of the differential equation gives

$$s^2 Y(s) - s + 1 - 9Y(s) = \frac{e^{-2s} - e^{-4s}}{s}.$$

Solving for $Y(s)$ leads to

$$Y(s) = \left(\frac{e^{-2s} - e^{-4s}}{s} + s - 1 \right) \left(\frac{1}{s^2 - 9} \right).$$

We can rewrite this as

$$Y(s) = (e^{-2s} - e^{-4s}) \left(\frac{1}{s(s^2 - 9)} \right) + \frac{s - 1}{s^2 - 9}.$$

Using partial fractions, we decompose

$$\frac{1}{s(s^2 - 9)} = \frac{A}{s} + \frac{B}{s - 3} + \frac{C}{s + 3},$$

where we find

$$A = -\frac{1}{9}, \quad b = \frac{1}{18}, \quad C = \frac{1}{18},$$

and

$$\frac{s - 1}{s^2 - 9} = \frac{A}{s - 3} + \frac{B}{s + 3},$$

where we find

$$A = \frac{1}{3}, \quad b = \frac{2}{3}.$$

Now we have

$$Y(s) = (e^{-2s} - e^{-4s}) \left(-\frac{1}{9s} + \frac{1}{18} \frac{1}{s - 3} + \frac{1}{18} \frac{1}{s + 3} \right) + \frac{1}{3} \frac{1}{s - 3} + \frac{2}{3} \frac{1}{s + 3}.$$

Now use the property

$$L[f(t - c)H(t - c)] = e^{-cs}F(s),$$

to find

$$\begin{aligned} y(t) &= \left(-\frac{1}{9} + \frac{1}{18}e^{3(t-2)} + \frac{1}{18}e^{-3(t-2)} \right) H(t - 2) - \\ &\quad \left(-\frac{1}{9} + \frac{1}{18}e^{3(t-4)} + \frac{1}{18}e^{-3(t-4)} \right) H(t - 4) + \\ &\quad \frac{1}{3}e^{3t} + \frac{2}{3}e^{-3t}. \end{aligned}$$