MATH495.3 (CRN 13695)

Laboratory # 8: Sensitivity analysis

1 Part 1: Background

Last week we looked at one model analysis tool called R_0. The concept has its pros and cons, but by itself tells us little of *how* to control an infection. If we know $R_0 > 1$, we expect the infection to persist, but is there anything we can do about it? Consider more general models that involve the dynamics of a certain species. If the trajectories predict extinction for the species, is there anything we can do to change the outcome? Sensitivity analysis is one tool that helps us answer these questions.

Recall last week's model of infection with the initial conditions.

$$\left.\begin{aligned} \frac{dS}{dt} &= -\beta SI + \delta(I+R) + \mu I, \\ \frac{dI}{dt} &= \beta SI - \gamma I - \mu I - \delta I \\ \frac{dR}{dt} &= \gamma I - \delta R \\ \\ [S(0), I(0), R(0)] &= [599, 1, 0] \end{aligned}\right\} \tag{1}$$

Again, we will use the following values for the parameters.

Parameter	Numerical Value	Interpretation
β	0.008	Infection rate
δ	0.1	Natural death rate
γ	0.3	Recovery rate
μ	0.1	Disease-specific death rate

Table 1: Parameter Values for SIR Model, units of β, δ, γ, and μ are $(\text{years})^{-1}$.

The model may be expressed in a more general form,

$$\left.\begin{aligned} \frac{d\boldsymbol{x}(t,\boldsymbol{p})}{dt} &= \boldsymbol{h}\big(\boldsymbol{x}(t,\boldsymbol{p}),\ \boldsymbol{p}\big) \\ \boldsymbol{x}(0) &= \boldsymbol{z} \end{aligned}\right\} \tag{2}$$

where $\boldsymbol{x} \in \mathbb{R}^{\mathrm{M}}, \boldsymbol{p} \in \mathbb{R}^{\mathrm{K}}$, and the initial conditions $\boldsymbol{z} \in \mathbb{R}^{\mathrm{M}}$. In our example, $\mathrm{M} = 3$, $\boldsymbol{x}(t,\boldsymbol{p}) = [S,I,R]$, $\mathrm{K} = 4$, $\boldsymbol{p} = [\beta,\delta,\gamma,\mu]$, and $\boldsymbol{z} = [599,1,0]$. Writing this vector equation using index notation,

$$\left.\begin{aligned} \frac{dx_i(t,\boldsymbol{p})}{dt} &= h_i(\boldsymbol{x}(t,\boldsymbol{p}),\boldsymbol{p}) \\ x_i(0) &= z_i \end{aligned}\right\}, \quad i = 1,\ldots,\mathrm{M}. \tag{3}$$

We wish to know which parameter influences the outcomes of the model the most. That is, if I change that parameter by a small amount, the solution will change the most. We'll define this concept in terms of derivatives. Define the sensitivity of state x_i to parameter p_j as

$$S_{i,j} = \frac{\partial x_i}{\partial p_j}. \tag{4}$$

The sensitivities for ODEs can be obtained by differentiating equation (3) with respect to the j^{th} parameter p_j, and then reversing the order of differentiation on the left-hand side.

$$\begin{aligned} \frac{\partial}{\partial p_j}\left(\frac{dx_i}{dt}\right) &= \left(\sum_{m=1}^{\mathrm{M}} \frac{\partial h_i}{\partial x_m}\frac{\partial x_m}{\partial p_j}\right) + \frac{\partial h_i}{\partial p_j} \qquad i = 1,\ldots,\mathrm{M},\ j = 1,\ldots \mathrm{K} \\ \frac{d}{dt}\left(\frac{\partial x_i}{\partial p_j}\right) &= \left(\sum_{m=1}^{\mathrm{M}} \frac{\partial h_i}{\partial x_m}\frac{\partial x_m}{\partial p_j}\right) + \frac{\partial h_i}{\partial p_j} \qquad i = 1,\ldots,\mathrm{M},\ j = 1,\ldots \mathrm{K} \end{aligned} \tag{5}$$

Equation (5) describes the sensitivities of states x_i to parameters p_j as ordinary differential equations. These equations must be solved with the model differential equations for x_i simultaneously, as both the states $x_i(t)$ and the sensitivities $\frac{\partial x_i}{\partial p_j}(t)$ depend on time.

2 Assignment

Question One At equilibrium, the sensitivities of infected individuals I to parameters $p_1, \ldots p_4$ are given in Table 2.

Parameter	Sensitivity of $I(t)$
β	1953
δ	976.5
γ	-367.1
μ	-31.25

Table 2: Sensitivities of the number if of infected individuals at equilibrium.

Let's see if these answers make sense. Answer the following questions for the analysis.

1. From your code "sir.m" last week, change the value of β by 10%. First, increase β by 10% and then decrease β by 10%. What happened to the equilibrium number of infected individuals when these changes were made? Repeat the results for every parameter, one at a time, returning the other parameters to their original value. You may want to allow the parameters to be inputs to the function sir($[\beta, \delta, \gamma, \mu]$) and loop over the function for the different inputs. Please make a table that has the following form:

Parameter	$I^\star$ after increase of p_j	$I^\star$ after decrease of p_j
β		
δ		
γ		
μ		

2. Why are the sensitivities of β and δ positive, while the sensitivities of γ and μ are negative?
3. Generally, the higher the sensitivity of a quantity of interest to a parameter, the more impact a change of that parameter will have on the quantity of interest. Is this consistent with what you found in your

table? Which parameter is inconsistent with this result? Make some sense of this– why might the sensitivity of this parameter p_j be so high and yet have little effect on $I^\star$?

Question Two Recall the formula for R_0.

$$R_0 = \frac{\beta S(0)}{\delta + \gamma + \mu} \tag{6}$$

Since R_0 does not depend on time, the sensitivity can be computed without using a differential equation. What is the sensitivity of R_0 to each parameter? Make another table with the sensitivities of R_0 to parameters and $S(0)$, using the values in Table 1 and initial conditions from (1):

Parameter	Sensitivity of R_0
β	
δ	
γ	
μ	
$S(0)$	

Which parameter (including $S(0)$) is R_0 most sensitive to? Is this similar to the sensitivity results in Table 2?