Stability of the O(2)-symmetric flow past a sphere in a pipe

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We examine the linear stability of the O(2)-symmetric laminar flow past a centrally-located sphere in a pipe for a range of sphere sizes. For all values of the blockage ratio studied, the first instability observed as the flow rate increased, occurred at a *steady* symmetry-breaking bifurcation point. A transformation technique was used to solve the generalized eigenvalue problem which arises when using a mixed finite-element method to determine linear stability. Our results support the conclusion of Natarajan & Acrivos[1] who have computed the stability of the flow past a sphere in an unbounded domain. The primary instability in three-dimensions differs qualitatively from that of the analogous two-dimensional flow past a centrally-located cylinder in a channel, which Chen, Pritchard & Tavener[2] have shown to occur at a Z_2-symmetry breaking Hopf bifurcation point.

1. Introduction

A recent paper by Natarajan & Acrivos[1] concludes that the low Reynolds number flow past a sphere in an unbounded domain becomes unstable at a steady bifurcation point as the Reynolds number is increased, and that the resulting asymmetric flow is therefore time-independent. Their result is surprising since it contradicts the previously published computational work of Kim & Pearlstein[3] who contend that the primary bifurcation is of Hopf type, and that the resulting asymmetric flow is time-dependent. Both pairs of authors show the first unstable mode to have an azimuthal wavenumber of one. The critical Reynolds numbers predicted by Natarajan & Acrivos[1] and by Kim & Pearlstein[3] are 210 and 175.1 respectively, when based on the sphere diameter and the uniform upstream velocity.

The experimental literature is also contradictory. Möller[4] used a system of pulleys and external weights to tow a sphere vertically through a tank with square cross-section. He observed the flow to be steady at a Reynolds number of 170, and to oscillate with low frequency in a region confined near the rear of the sphere at a Reynolds number of 200. Taneda[5] used a motor and lead screw system to drag a sphere horizontally through a rectangular water tank. He reported periodic oscillations in the otherwise axisymmetric wake region above a Reynolds number of 130, even though vortex shedding did not occur until the Reynolds number exceeded 300. Roos & Willmarth[6] towed spheres through water/glycerin mixtures and observed unsteadiness in the wake at a Reynolds number of approximately 290 when the sphere was supported by a rear mounted sting, and at a Reynolds number of 215 when the sphere was supported on two fine threads. These authors measured both the drag and the side force on the sphere, and should therefore have been able to detect any non-zero side forces had they existed before unsteadiness was observed.

Goldburg & Florsheim[7], Zikmundova[8] and Nakamura[9] all performed experiments in which spheres coated with dye were observed as they fell through a quiescent body of water or water/glycerin solutions. In earlier work, Magarvey & Bishop[10] observed the fall of a droplet of carbon tetrachloride or chlorobenzene in water. Magarvey & Bishop[10] observed the wake behind the droplet to be a single axisymmetric thread for Reynolds numbers less than 210. Within the Reynolds number range 210–270, the vortex region behind the droplet was clearly non-axisymmetric and two threads of dye were left behind the drop, which no longer fell vertically. For Reynolds numbers above this range, the wake was time dependent. Goldburg & Florsheim[7] were able to reproduce all these observations using solid spheres, although they found the onset of the steady asymmetric wake to occur at a slightly lower Reynolds number. They found the same critical value of Re=270 for the onset of time-dependent motion. This agreement is surprising, since for these two problems to be equivalent, the liquid drop must be spherical and the interface between the two fluids must act as a nonslip surface. Both assumptions seem doubtful without convincing evidence. Further, Sakamoto & Haniu[11] observe that the Strouhal numbers reported by Magarvey & Bishop[10] for time-dependent flows between Reynolds numbers 300 and 1000, differ considerably from those of other investigators using solid spheres.

Zikmundova[7] found that for a Reynolds number near 110, a "slight asymmetry" appeared in the wake. When the attachment angle of the recirculation region (on the surface of the sphere) was plotted against the Reynolds number, the shape of the curve was found to change dramatically at a Reynolds number between 130 and 150, which phenomena the author attributed to the onset of a Karman vortex street. Unfortunately the tank in which these experiments were performed was extremely short, raising doubts as to whether the spheres attained their terminal velocity. Nakamura[8] found a Reynolds number of 190 to be the upper limit for a stable axisymmetric ring eddy behind the sphere. For Reynolds numbers exceeding 190 the recirculation region was steady but clearly asymmetric, and the sphere no longer fell vertically.

Most recently, Wu & Faeth[12] towed spheres vertically through water / glycerin mixtures in a rectangular tank. They supported the sphere using a single taut wire passing through its center. The wake region was found to be steady and axisymmetric for Reynolds numbers less than 200, and to be steady but non-axisymmetric for Reynolds numbers between 200 and 280. For Reynolds numbers exceeding 280 the wake was unsteady with vortex shedding.

The flows examined here are enclosed in a pipe, as we believe these problems to be better defined than the flow in an unbounded domain. Following the work of Amick[13], it can be shown that for small enough Reynolds numbers, the flow of a Newtonian fluid past an obstruction in a straight pipe will decay exponentially to the unperturbed Poiseuille flow both up and downstream of the obstruction. Since laboratory experiments must be carried out in a finite domain, such exponential decay ensures that an experiment closely matching the unbounded problem can be performed provided a sufficient length of pipe is placed both up- and downstream of the sphere. Experiments employing this principle are reported in Pritchard[14] who made some measurements of the drag on the body. Similarly, finite-element computations must be performed on a finite domain, and the exponential decay ensures that numerical calculations which closely approximate the problem on the unbounded domain can be performed provided the computational domain is long enough. There are no equivalent results in an unbounded domain.

A knowledge of the instability mechanisms which exist in other simple laminar flows does not provide any intuition for the expected behaviour of the flow past a sphere. The analogous two-dimensional problem, which is to determine the linear stability of the Z_2-symmetric laminar flow past a centrally-located cylinder in a channel, has been studied by Chen, Pritchard & Tavener[2]. (The low Reynolds number flow past the cylinder is said to be Z_2-symmetric since the velocity component parallel to the channel walls and the pressure are invariant under reflection about the centerline of the channel, while the sign of the velocity component perpendicular to the channel walls is reversed under reflection about the centerline of the channel.) They show that the primary flow loses stability as the Reynolds number is increased at a Z_2-symmetry breaking Hopf bifurcation point. The instability mechanism was qualitatively similar for all blockage ratios (cylinder diameter to channel width) they examined, within the range 0.1 to 0.7. To determine the nature of the instability they performed time-dependent computations using the commercial code FIDAP[15], as well as direct computation of the Hopf bifurcation point using the finite-element code ENTWIFE[16], the same code as is used in the present study. This code implements the method of Griewank & Reddien[17] which enables the Reynolds number at a Hopf bifurcation point, as well as the critical frequency and the real and imaginary components of the null eigenvector, to be found as the solution of an extended system of equations. An initial guess for the imaginary part of the null eigenvalue, and for the complex null eigenvector, was found by solving the generalized eigenvalue problem arising from a linear stability analysis. This eigenvalue problem was solved using the Cayley transform method of Cliffe, Garratt & Spence[18], which is also employed in the present study. Its application to O(2)-symmetric flows is discussed in Section 3 and the appendix. The flow past a cylinder in an unbounded domain had been previously found to lose stability via Hopf bifurcation by Jackson[19] and Zebib[20].

The two-dimensional flow in a channel with a sudden symmetric 1:3 expansion has been shown by Fearn, Mullin & Cliffe[21] to undergo a Z_2-symmetry breaking *steady* bifurcation at a Reynolds number of approximately 54, when the Reynolds number is based on the mean flow and upstream channel width. The ability to detect this steady instability was one of the first tests of the Cayley transform technique. The pair of asymmetric flows which bifurcate supercritically from the pitchfork bifurcation point, has been shown by this technique to be stable with respect to two-dimensional disturbances at a Reynolds number for which time-dependent behaviour is observed experimentally by Fearn, Mullin & Cliffe[21]. The first time dependent flows are apparently three-dimensional in character. Similar behaviour is observed in the flow over a backward facing step which, as computed by Gresho et. al.[22], is stable with respect to two-dimensional disturbances for a Reynolds number at which time-dependence is observed experimentally. Work in progress by this author investigating the linear stability of the flow in pipes which undergo a sudden symmetric expansion, suggests that the first instability of the O(2)-symmetric laminar flow occurs at a Hopf bifurcation point with azimuthal wavenumber one. For this problem therefore, the flows which bifurcate first from the steady low Reynolds number flow are time dependent.

The computations reported here of the O(2)-symmetric laminar flow past a sphere in a pipe were performed for blockage ratios from 0.2 to 0.7, where the blockage ratio b is defined as the ratio of the sphere diameter to the pipe diameter. (The low Reynolds number flow past a centrally-located sphere in a pipe is said to be $O(2)$-symmetric since the velocities and pressure, when measured in a cylindrical coordinate system whose axis is aligned with the centerline of the pipe, are invariant under any rotation about the axis, and under reflection in any diameter of the pipe. This reflectional symmetry means that O(2)-symmetric flows are without swirl.) In all cases the primary bifurcation was steady with an azimuthal wavenumber of one, and the velocity components of the null eigenvector were similar to those reported by Natarajan & Acrivos[1]. Given the computational difficulties encountered at blockage ratios smaller than 0.2, it was not possible to make an accurate quantitative prediction of the critical Reynolds number at zero blockage ratio (in which limit the flow studied here approaches that in an unbounded domain), but our results strongly support the computations of Natarajan & Acrivos[1] which show the primary bifurcation in an unbounded domain to be steady and the null eigenvector to have an azimuthal wavenumber of one.

2. The governing equations

Let Ω be a bounded domain with a piecewise smooth boundary , . The domain for the present study is shown in Fig. 1, and is a radial slice through a pipe with a centrally-located sphere. Let $,_n$ and $,_\tau$ to be two segments of the boundary , , which are not necessarily distinct, and whose union need not comprise the entire boundary. We wish to find the velocity and pressure fields $\boldsymbol{u}(\boldsymbol{x}, t)$ and $p(\boldsymbol{x}, t)$ respectively, which satisfy the incompressible Navier-Stokes equations

$$R\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) \;=\; -\nabla p + \Delta\boldsymbol{u} \;, \qquad (1)$$

and

$$\nabla\cdot\boldsymbol{u} \;=\; 0 \;, \qquad (2)$$

in Ω, subject to the boundary conditions

$$\boldsymbol{u}\cdot\boldsymbol{n} \;= g_n \text{ on } ,_n \;, \qquad (3)$$

and

$$\boldsymbol{n}\times\boldsymbol{u}\times\boldsymbol{n} \;= \boldsymbol{g}_\tau \text{ on } ,_\tau \;, \qquad (4)$$

where $\boldsymbol{n}$ and $\boldsymbol{t}$ are the unit normal and unit tangent vectors respectively. The Navier-Stokes equations in cylindrical coordinates (the appropriate coordinate system for the problem studied here), are given as equations (18)–(21) in the appendix.

In order to nondimensionalize the Navier-Stokes equations as presented in equations (1)–(4), appropriate length and velocity scales must be defined. The diameter of the pipe D was chosen as the representative length scale, and the mean velocity U was chosen as the velocity scale. Here

$$U = \frac{4}{\pi D^2}\int_0^{D/2} 2\pi r u_z(r)\; dr \;,$$

where $u_z(r)$ is the axial velocity at the upstream boundary of Ω. The Reynolds number was then $R = DU/\nu$. For the purposes of comparison with computations on unbounded domains, a second Reynolds number R_d was defined, based on the sphere diameter d, and the velocity averaged across the sphere cross-section, U_d. Hence $R_d = dU_d/\nu$, where

$$U_d = \frac{4}{\pi d^2} \int_0^{d/2} 2\pi r u_z(r)\, dr \ .$$

Dirichlet velocity boundary conditions were applied on the normal and tangential velocity components at the inlet boundary, along the pipe wall and on the surface of the sphere. A parabolic axial velocity, and zero radial and azimuthal velocities were imposed at the inlet, and non-slip conditions were imposed along the pipe wall and on the sphere. When computing the velocity and pressure field of the O(2)-symmetric flow but not considering its linear stability, the radial velocity along the centerline was set to zero and natural boundary conditions were applied to the tangential velocity components. The Dirichlet condition on the radial velocity along the centerline was replaced by the natural boundary condition when performing stability calculations for reasons outlined below. Natural boundary conditions were always imposed at the outlet.

Let $(\boldsymbol{u}, p)$ be a steady solution of (1)–(2), such that $\boldsymbol{u} \in V_g$ and $p \in L_0^2$, where

$$V_g = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v} \cdot \boldsymbol{n} = g_n \text{ on } ,_n, \boldsymbol{n} \times \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{g}_\tau \text{ on } ,_\tau\} \ ,$$

and

$$L_0^2 = \left\{ q \in L^2(\Omega) : \int_\Omega q \, \mathrm{d}\Omega = 0 \right\} .$$

Here $\boldsymbol{H}^1(\Omega)$ is the space of vector functions defined on Ω whose function values and first derivatives lie in $L^2(\Omega)$.

To construct the weak form of the steady Navier-Stokes equations (see e.g. Gunzburger[23]) we take the dot product of (1) with test functions $\boldsymbol{v} \in V_0$ where

$$V_0 = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } ,_n, \boldsymbol{n} \times \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } ,_\tau\} \ ,$$

and the scalar product of (2) with test functions $q \in L_0^2$, then integrate over the domain Ω. Integrating by parts where appropriate we have

$$R \int_\Omega (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \cdot \boldsymbol{v} \, dV - \int_\Omega p(\nabla \cdot \boldsymbol{v}) \, dV + \int_\Gamma p(\boldsymbol{v} \cdot \boldsymbol{n}) \, dS$$

$$+\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v}\, dV - \int_\Gamma (\boldsymbol{v}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{n}\, dS \quad = \quad 0\ , \tag{5}$$

and

$$\int_\Omega q(\nabla\cdot\boldsymbol{v})\, dV \quad = \quad 0\ . \tag{6}$$

A weak solution of the Navier-Stokes equations is determined by requiring the integrals over the domain Ω to vanish, i.e. by solving

$$R\int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v}\, dV - \int_\Omega p(\nabla\cdot\boldsymbol{v})\, dV + \int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{v}\, dV \quad = \quad 0\ ,$$

and

$$\int_\Omega q(\nabla\cdot\boldsymbol{v})\, dV \quad = \quad 0\ .$$

It is assumed that the integrals over the boundary , also vanish, and the so-called natural boundary conditions which are thereby implied are

$$p - \boldsymbol{n}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{n} \quad = \quad 0 \text{ on } ,\ \backslash ,\ _n\ , \tag{7}$$

and

$$\boldsymbol{n}\cdot\nabla\boldsymbol{u}\times\boldsymbol{n} \quad = \quad \boldsymbol{0} \text{ on } ,\ \backslash ,\ _\tau\ . \tag{8}$$

Following Benjamin[24], we will use the term "primary" flow to mean the flow that is stable for very small Reynolds number and whose linear stability or instability we seek to determine as the flow rate increases. The primary flow is both time-independent and O(2)-symmetric, and therefore the azimuthal velocity component of the primary flow is zero throughout Ω, as are the azimuthal derivatives of the axial and radial velocity components, and the azimuthal derivative of the pressure.

In order to determine the linear stability of a steady solution of the Navier-Stokes equations $(\boldsymbol{u}_0, p_0)$ say, we consider solutions of the form

$$(\boldsymbol{u}_0, p_0) + \epsilon\Big(\hat{\boldsymbol{u}}_1 \exp(-\gamma t), \hat{p}_1 \exp(-\gamma t)\Big), \tag{9}$$

where $0 < \epsilon << 1$ and $\gamma = \mu + i\omega$ is complex. Here we will look for

$$\hat{\boldsymbol{u}}_1 = \boldsymbol{u}_1(r,z)\exp(im\theta), \qquad \hat{p}_1 = p_1(r,z)\exp(im\theta) \tag{10}$$

with $\boldsymbol{u}_1 \in V_0,\ p_1 \in L_0^2$, and m is an integer. Substituting (9) into the Navier-Stokes equations and collecting terms of order ϵ, the equations governing the perturbation are

$$R\Big[-\gamma\hat{\boldsymbol{u}}_1 + (\boldsymbol{u}_0 \cdot \nabla)\hat{\boldsymbol{u}}_1 + (\hat{\boldsymbol{u}}_1 \cdot \nabla)\boldsymbol{u}_0\Big] \;=\; -\nabla\hat{p}_1 + \Delta\hat{\boldsymbol{u}}_1\ , \tag{11}$$

and

$$\nabla \cdot \hat{\boldsymbol{u}}_1 \;=\; 0\ . \tag{12}$$

The linearized equations in cylindrical coordinates for an O(2)-symmetric steady solution $(\boldsymbol{u}_0, p_0)$, and perturbations defined by (10), are given in the appendix as equations (23)–(26).

We construct a weak form of the linearized equations assuming disturbances of the form (10) by taking the dot product of (11) with $\boldsymbol{v}_1 \in V_0$ and the scalar product of (12) with $q_1 \in L_0^2$, then integrating over the domain Ω. Integrating by parts where appropriate we have

$$\begin{aligned} R\Big[\int_\Omega -\gamma(\hat{\boldsymbol{u}}_1 \cdot \boldsymbol{v}_1)\ dV + \int_\Omega \Big((\boldsymbol{u}_0 \cdot \nabla)\hat{\boldsymbol{u}}_1 + (\hat{\boldsymbol{u}}_1 \cdot \nabla)\boldsymbol{u}_0\Big) \cdot \boldsymbol{v}_1\ dV \quad \Big] \\ - \int_\Omega \hat{p}_1(\nabla \cdot \boldsymbol{v}_1)\ dV + \int_\Gamma \hat{p}_1(\boldsymbol{v}_1 \cdot \boldsymbol{n})\ dS \\ + \int_\Omega \nabla\hat{\boldsymbol{u}}_1 : \nabla\boldsymbol{v}_1\ dV - \int_\Gamma (\boldsymbol{v}_1 \cdot \nabla\hat{\boldsymbol{u}}_1) \cdot \boldsymbol{n}\ dS \;&=\; 0, \end{aligned} \tag{13}$$

and

$$\int_\Omega q_1(\nabla \cdot \hat{\boldsymbol{u}}_1)\ dV \;=\; 0. \tag{14}$$

If we again solve these equations by requiring the integrals over the domain to vanish and assuming that the integrals over the boundary are zero, the natural boundary conditions for the perturbation velocities $\boldsymbol{u}_1$ and perturbation pressures p_1, for any integer wavenumber m, are exactly the same as those for the solution velocities $\boldsymbol{u}_0$ and pressures p_0, which are given by equations (7) and (8). Dirichlet velocity boundary conditions can not be set on the normal velocity component along the centerline for stability calculations, as this would require the normal velocity component of the perturbation to be zero along the centerline, over-constraining the set of permissible disturbances. The weak forms of the linearized equations in cylindrical coordinates about an O(2)-symmetric steady solution $(\boldsymbol{u}_0, p_0)$, and perturbations defined by (10), are given in the appendix as equations (27)–(30).

Discrete mixed methods based on these variational formulations are derived by choosing finite-dimensional spaces $V_{h,g} \subset V_g$, $V_{h,0} \subset V_0$ and $\Pi_h \subset L_0^2$ appropriately, where the parameter h measures the fineness of the discretization. Isoparametric quadrilateral elements were used with biquadratic interpolation of the velocity field and discontinuous linear interpolation of the pressure field. The integrals were evaluated on each quadrilateral using a 9-point Gaussian quadrature (at interior points) then summed over all elements.

Let $\boldsymbol{w} \in \mathrm{R}^k$ be the vector of nodal freedoms defining the velocity perturbation $\boldsymbol{u}_1$, and $\boldsymbol{p} \in \mathrm{R}^l$ be the vector of nodal freedoms defining the pressure perturbation p_1. For an O(2)-symmetric steady solution and a given integer wavenumber m, the discrete form of (13) and (14) is a generalized eigenvalue problem of the form

$$\begin{bmatrix} K & C \\ C^T & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{p} \end{pmatrix} = \gamma \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{p} \end{pmatrix}, \qquad (15)$$

where $K \in \mathrm{R}^{k \times k}$ is sparse and nonsymmetric of rank k, $C \in \mathrm{R}^{k \times l}$ has rank l, and $M \in \mathrm{R}^{k \times k}$ is symmetric positive definite.

3. Solution of the eigenvalue problem

The linear stability argument of the previous section has produced a generalized eigenvalue problem $A\boldsymbol{z} = \gamma B\boldsymbol{z}$, where matrices A and B have the block structure illustrated in equation (15), and $\boldsymbol{z}^T = (\boldsymbol{w}^T, \boldsymbol{p}^T)$, $\boldsymbol{z} \in \mathrm{R}^N$ where $N = k + l$. The matrix B is clearly singular, a consequence of the fact that the continuity equation is time independent, and the matrix pair (A, B) has only $(k-l)$ finite eigenvalues γ_i, $i = 1, \cdots, k-l$. The $2l$ missing or "infinite" eigenvalues, are zero eigenvalues of the matrix pair (B, A), i.e. zero eigenvalues of $B\boldsymbol{z} = \nu A\boldsymbol{z}$. Since A and B are real matrices when the primary flow is O(2)-symmetric (as discussed in the appendix), the eigenvalues γ_i, $i = 1, \cdots, k-l$ are either real or exist as complex conjugate pairs.

A solution of the discretized steady Navier-Stokes equations is linearly stable if and only if all eigenvalues of (15) lie in the right-half of the complex plane. For sufficiently small Reynolds number the O(2)-symmetric primary flow past a sphere in a pipe is stable. The Reynolds number at which this flow loses stability may therefore be determined by finding the first eigenvalue(s) to cross from the right-half to the left-half plane as the Reynolds number is increased. Hence only the eigenvalues of (15) with smallest real part are of interest. In order to find these so-called "most dangerous eigenvalues", we have used the modified Cayley transform technique of Cliffe, Garratt & Spence[18] which constructs a new generalized eigenvalue problem

$$\begin{pmatrix} K-\alpha_2 M & \alpha_3 C \\ \alpha_3 C^T & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{s} \end{pmatrix} = \theta \begin{pmatrix} K-\alpha_1 M & C \\ C^T & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{s} \end{pmatrix}, \tag{16}$$

where $\boldsymbol{w} \in \mathrm{R}^k, \boldsymbol{s} \in \mathrm{R}^l$, $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{R}$, $\alpha_1 \neq \gamma_i,\ i = 1, \cdots, k-l$ and $\alpha_1 < \alpha_2$. The eigenvalue problem (16) has $(k-l)$ eigenvalues

$$\theta_i = \frac{\gamma_i - \alpha_2}{\gamma_i - \alpha_1}, \quad i = 1, \cdots, k-l, \tag{17}$$

corresponding to the $(k-l)$ finite eigenvalues $\gamma_i,\ i = 1, \cdots, k-l$ of (15), and $2l$ eigenvalues equal to α_3, corresponding to the $2l$ "infinite" eigenvalues of (15). By comparing the distances between an arbitrary point γ_i in the complex plane, and the two real points α_1 and α_2, it is easy to see from (17) that $|\theta_i| < 1$ when the real part of γ_i, $\mathrm{Re}(\gamma_i) = \mu_i > \frac{1}{2}(\alpha_1 + \alpha_2)$, and $|\theta_i| > 1$ when $\mu_i < \frac{1}{2}(\alpha_1 + \alpha_2)$. If equation (17) is considered as a mapping from the γ-plane to the θ-plane, it is seen to map all points in the γ-plane to the left of the line $\mu = \frac{1}{2}(\alpha_1 + \alpha_2)$ to points outside the unit circle in the θ-plane, and to map all points in the γ-plane to the right of the line $\mu = \frac{1}{2}(\alpha_1 + \alpha_2)$, to points inside the unit circle in the θ-plane. Eigenvalues of (15) with real part less than $\frac{1}{2}(\alpha_1 + \alpha_2)$ can therefore be found by setting $\alpha_3 = 0$ and applying subspace iteration to (16). The first k (velocity) components of the corresponding eigenvectors of (15) and (16) are equal, while the remaining l (pressure) components are related in a known manner. An iterative strategy for selecting α_1 and α_2 was adopted.

4. Numerical results

In all cases discussed below, the axisymmetric primary solution and the generalized eigenvalue problem arising from the linear stability analysis were computed using the same finite-element grid. Considerable care was taken to ensure that both the trivial solution and the critical Reynolds number obtained were grid independent over the range of blockage ratios studied.

Stability computations were first performed for a blockage ratio of 0.5 using a number of domains of different lengths and a variety of discretizations. The shortest domain and coarsest discretization on which computations were performed at this blockage ratio is shown in Fig. 2. This mesh extends 3 pipe diameters upstream of the center of the sphere and $7\frac{1}{2}$ pipe diameters downstream, and comprises 300 quadrilateral elements with biquadratic velocity interpolation and discontinuous linear pressure interpolation on each element. The longest domain on which computations were performed extended 25 pipe diameters downstream of the center of the sphere. Meshes with at least three times the number of elements of the coarse mesh illustrated, and having 13584 degrees of freedom, were used to compute the quantities plotted in Figs. 3–8.

For sufficiently small Reynolds number the steady, O(2)-symmetric primary flow was stable. The Reynolds number was increased and the linear stability calculation repeated until a single eigenvalue (or pair of eigenvalues) $\gamma = \mu + i\omega$ of the generalized eigenvalue problem (15), was observed to cross from the stable right-half of the complex plane into the unstable left-half plane.

For a blockage ratio of 0.5 and an azimuthal wavenumber m equal to one, a real eigenvalue was found to cross the imaginary axis first, at a Reynolds number (R_d) of approximately 334. The velocity components of the null eigenvector are plotted in Fig. 3 and closely resemble those appearing in Fig. 8(a), page 338 of Natarajan & Acrivos[1]. The only instance in which the initial instability appeared to be time-dependent was when, keeping the number of elements fixed, the domain downstream of the sphere was increased to such an extent that the downstream flow became very poorly resolved. Under these circumstances alone, a complex conjugate eigenvalue pair was observed to cross into the unstable left half plane; all other eigenvalues having positive real part. This behaviour was eliminated on domains of an equal length by increasing the number of elements downstream of the sphere. As further evidence that the apparent Hopf bifurcation was spurious, the velocity components of the (complex) null eigenvector were clearly inappropriate.

To ensure that the $m = 1$ instability was the crucial instability for a blockage ratio of 0.5, the most dangerous eigenvalues $\gamma = \mu + i\omega$ of (15) at a Reynolds number $R_d = 350$ were computed for azimuthal wavenumbers zero to four. The real and imaginary parts of these eigenvalues are plotted against the azimuthal wavenumber m in Fig. 4. At this Reynolds number the only unstable mode was that with wavenumber one, while the other modes were linearly stable. For wavenumbers zero and one, the most dangerous eigenvalue was found to be real, while for wavenumbers two, three and four, the most dangerous eigenvalues were complex conjugate pairs with increasing real and imaginary parts. This is consistent with the data presented in figure 10(a), page 340 of Natarajan & Acrivos[1].

The entire procedure was then repeated for a range blockage ratios between 0.2 and 0.7. The critical Reynolds number R_d is shown as a function of blockage ratio in Fig. 5. For all values of the blockage ratio studied, the first bifurcation was found to be steady, and the null eigenvector to have an azimuthal wavenumber of one. The three velocity components of the $m = 1$ null eigenvector were similar for all blockage ratios. Mesh refinement studies at blockage ratios 0.2, 0.3, 0.5 and 0.7 have shown the critical Reynolds numbers to have converged to within 1% on a mesh with 13584 degrees of freedom. Studies were also conducted to ensure that the most dangerous eigenvalues reported were independent of the dimension of the subspace and number of orthogonal iterations used in the subspace iteration. A subspace of dimension 30 with 30 orthogonal iterations was found to be more than adequate in all cases.

Computations were attempted at a blockage ratio of 0.1, but convergence to within 1% could not be guaranteed for this case with the computational resources available, and so the results of these calculations are not recorded in Fig. 5. For a blockage ratio of 0.1, the ratio of the cross-sectional area of the sphere to the cross-sectional area of the the pipe is only 0.01, and the design of finite-element grids to adequately resolve the flow near the sphere is very challenging. For this reason a quantitative comparison with the finite-difference computations of Johansson[25] was not attempted as he presents results for a blockage ratio of 0.1 only. The imperfect computational evidence suggests a steady bifurcation with an azimuthal wavenumber of 1. Our best estimate of the critical Reynolds number, R_d for this blockage ratio is 225.

The computations of Chen, Pritchard & Tavener[2] illustrate a similar relationship between blockage ratio and the critical Reynolds number for flow past a cylinder in a channel, when the Reynolds number is based on the cylinder diameter. In both cases, the critical Reynolds number has a maximum near a blockage ratio of 0.5. This maximum is not believed to be particularly significant, but to be a consequence of the changing length scale. Evidence for this contention will be discussed below.

The values of R_d shown in Fig. 5 should be compared with a critical Reynolds number of 210 (based on the sphere diameter and the uniform upstream velocity), predicted by Natarajan & Acrivos[1] for flow past a sphere in an unbounded domain, and supported by much of the experimental evidence as discussed in the introduction. The flow in an unbounded domain may be considered to be the limiting case of zero blockage ratio for the problems studied here. The difficulties experienced at blockage ratios smaller than 0.2 make any prediction of the critical Reynolds number zero blockage ratio case tenuous. Even if a critical Reynolds number at a smaller blockage ratio, say 0.1, were known, it is still unclear how to extrapolate to the zero blockage ratio case. For example, using a value of $R_d = 225$ at a blockage ratio of 0.1 and fitting polynomials of increasing degree to the first two, three and four data points, leads to a wide scattering of the predicted values at $b = 0$.

The critical values of the Reynolds number R based on the pipe diameter, are plotted against the blockage ratio in Fig. 6. As the blockage ratio decreases the problem approaches that of flow in an unblocked pipe which is linearly stable for all Reynolds numbers. An increase in the critical Reynolds number as the blockage ratio decreases is therefore to be expected. This increase is more dramatic when the estimated value at a blockage ratio of 0.1 is included. Using this definition of the Reynolds number, the behaviour near a blockage ratio of 0.5 is not exceptional.

To provide some further quantities for experimental comparison, and to reinforce our conclusion that the instability mechanism is qualitatively similar for all blockage ratios examined, the length of the recirculation region at the critical Reynolds number is plotted as a function of blockage ratio in Fig. 7. The length of the recirculation region is defined to be the distance along the centerline from the rear of the sphere to the stagnation point in the wake. This length may be nondimensionalized with respect to either the sphere diameter, which value we call l_d, or with respect to the the pipe diameter, which we call l_D. Values of both l_d and l_D are plotted in Fig. 7. The length of the recirculation regions was computed on a sequence of increasingly fine grids for blockage ratios 0.2, 0.3, 0.5 and 0.7. For these blockage ratios the values obtained on meshes with 13584 degrees of freedom have converged to better than 1%. Notice that the behaviour near a blockage ratio of 0.5 is unremarkable.

The length of the separated eddy is a standard means of comparing two sets of computations, but since the precise location of the zero streamline is difficult to determine in flow visualization studies, it may not be the best measure for the purposes of comparison with experiment. A second measure is provided in Fig. 8, where the nondimensionalized pressure drop along the pipe wall from one pipe diameter upstream of the center of the sphere to one pipe diameter downstream of the center of the sphere, is determined at the critical Reynolds number and plotted against the blockage ratio. The pressure drops were computed on a sequence of increasingly fine grids for blockage ratios 0.2, 0.3, 0.5 and 0.7. For these blockage ratios the values obtained on meshes with 13584 degrees have converged to better than 1%. Again, we note that the behaviour near a blockage ratio of 0.5 is reasonably smooth. As a reference point, the pressure drop along an equal length of unblocked pipe, i.e. the pressure drop resulting from Pousieuille flow along an equal length of pipe, is shown at a blockage ratio of 0. It was determined from the analytic Pousieuille flow solution and can also computed as a rather trivial check of the numerical method. It is independent of the Reynolds number.

No attempt was made to compute the three-dimensional steady non-axisymmetric flow which bifurcates from the primary flow at a Reynolds number R_1 say, nor to determine its stability. It seems reasonable, given the experimental evidence, to assume that the steady bifurcation on the primary symmetric flow is supercritical and that the asymmetric branch undergoes a Hopf bifurcation at a Reynolds number slightly greater than R_1. Three-dimensional time-dependent computations by Tomboulides, Orszag & Karniadakis[26] support this conjecture. Natarajan & Acrivos[1] suggest that the Reynolds number $R_2 > R_1$, at which the first conjugate eigenvalue pair of the unstable, symmetric flow crosses the imaginary axis, provides an estimate of the Reynolds number at which the asymmetric flow undergoes Hopf bifurcation. They are careful to point out that this is speculative and if true, depends upon the fact that the Hopf bifurcation along the asymmetric branch occurs at a Reynolds number very close to R_1. Indeed the computations described here and those of Natarajan & Acrivos[1], cannot eliminate the possibility that the steady bifurcation from the primary flow is subcritical, under which circumstances there may be an Hopf bifurcation point along the subcritical asymmetric branch at a Reynolds number less than R_1. The primary flow would then lose stability with respect to a developed periodic flow and would display hysteresis on reduction of the Reynolds number. Similar behaviour was observed both numerically and experimentally by Tavener, Mullin & Cliffe[27] in a variant of Taylor-Couette system with rotating end walls. However the consensus of the experimental literature, especially the more recent studies, makes this scenario highly unlikely.

5. Acknowledgments

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Appendix. Formulation in cylindrical coordinates

The Navier-Stokes equations which govern the motion of an incompressible, Newtonian fluid may be written in cylindrical coordinates as

$$
\begin{aligned}
&R\left(\tfrac{\partial u_r}{\partial t} + u_r \tfrac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\tfrac{\partial u_r}{\partial \theta} + u_z \tfrac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r}\right) \\
&\quad + \tfrac{\partial p}{\partial r} \\
&\quad - \left(\tfrac{1}{r}\tfrac{\partial}{\partial r}(r\tfrac{\partial u_r}{\partial r}) + \tfrac{1}{r^2}\tfrac{\partial^2 u_r}{\partial \theta^2} + \tfrac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\tfrac{\partial u_\theta}{\partial \theta}\right) = 0, \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
&R\left(\tfrac{\partial u_\theta}{\partial t} + u_r \tfrac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\tfrac{\partial u_\theta}{\partial \theta} + u_z \tfrac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r}\right) \\
&\quad + \tfrac{1}{r}\tfrac{\partial p}{\partial \theta} \\
&\quad - \left(\tfrac{1}{r}\tfrac{\partial}{\partial r}(r\tfrac{\partial u_\theta}{\partial r}) + \tfrac{1}{r^2}\tfrac{\partial^2 u_\theta}{\partial \theta^2} + \tfrac{\partial^2 u_\theta}{\partial z^2} + \frac{2}{r^2}\tfrac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right) = 0, \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
&R\left(\tfrac{\partial u_z}{\partial t} + u_r \tfrac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\tfrac{\partial u_z}{\partial \theta} + u_z \tfrac{\partial u_z}{\partial z}\right) \\
&\quad + \tfrac{\partial p}{\partial z} \\
&\quad - \left(\tfrac{1}{r}\tfrac{\partial}{\partial r}(r\tfrac{\partial u_z}{\partial r}) + \tfrac{1}{r^2}\tfrac{\partial^2 u_z}{\partial \theta^2} + \tfrac{\partial^2 u_z}{\partial z^2}\right) = 0, \qquad (20)
\end{aligned}
$$

$$
\tfrac{1}{r}\tfrac{\partial}{\partial r}(r u_r) + \tfrac{1}{r}\tfrac{\partial u_\theta}{\partial \theta} + \tfrac{\partial u_z}{\partial z} = 0. \qquad (21)
$$

Let $(u_r, u_\theta, u_z, p) = (u_r^0, 0, u_z^0, p^0)$ be a steady O(2)-symmetric solution of equations (18)–(21) in Ω such that $(u_r^0, 0, u_z^0) \in V_g$, $p^0 \in L_0^2$ and

$$
u_\theta = 0, \quad \tfrac{\partial}{\partial t}(u_r^0, u_z^0, p^0) = 0, \text{ and } \tfrac{\partial}{\partial \theta}(u_r^0, u_z^0, p^0) = 0.
$$

Consider a nearby solution of the form

$$
\begin{aligned}
&(u_r^0, 0, u_z^0, p^0) \\
&\quad + \epsilon\left(u_r^1(r,z)\exp(im\theta - \gamma t), u_\theta^1(r,z)\exp(im\theta - \gamma t),\right. \\
&\qquad\quad \left. u_z^1(r,z)\exp(im\theta - \gamma t), p^1(r,z)\exp(im\theta - \gamma t)\right), \qquad (22)
\end{aligned}
$$

where $0 < \epsilon << 1$, $(u_r^1, u_\theta^1, u_z^1) \in V_0$, $p^1 \in L_0^2$, m is a positive integer and $\gamma = \mu + i\omega$ is complex. Substituting the new solution into the equilibrium equations (18)–(21) and equating to zero terms that are first-order in ϵ, the linearized equations are

$$R\left(-\gamma u_r^1 + u_r^0\frac{\partial u_r^1}{\partial r} + u_r^1\frac{\partial u_r^0}{\partial r} + u_z^0\frac{\partial u_r^1}{\partial z} + u_z^1\frac{\partial u_r^0}{\partial z}\right) + \frac{\partial p^1}{\partial r} - \left(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_r^1}{\partial r}) - \frac{m^2}{r^2}u_r^1 + \frac{\partial^2 u_r^1}{\partial z^2} - \frac{u_r^1}{r^2} - \frac{2im}{r^2}u_\theta^1\right) = 0, \tag{23}$$

$$R\left(-\gamma u_\theta^1 + u_r^0\frac{\partial u_\theta^1}{\partial r} + u_z^0\frac{\partial u_\theta^1}{\partial z} + \frac{u_r^0 u_\theta^1}{r}\right) + \frac{im}{r}p^1 - \left(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_\theta^1}{\partial r}) - \frac{m^2}{r^2}u_\theta^1 + \frac{\partial^2 u_\theta^1}{\partial z^2} + \frac{2im}{r^2}u_r^1 - \frac{u_\theta^1}{r^2}\right) = 0, \tag{24}$$

$$R\left(-\gamma u_z^1 + u_r^0\frac{\partial u_z^1}{\partial r} + u_r^1\frac{\partial u_z^0}{\partial r} + u_z^0\frac{\partial u_z^1}{\partial z} + u_z^1\frac{\partial u_z^0}{\partial z}\right) + \frac{\partial p^1}{\partial z} - \left(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_z^1}{\partial r}) - \frac{m^2}{r^2}u_z^1 + \frac{\partial^2 u_z^1}{\partial z^2}\right) = 0, \tag{25}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r^1) + \frac{im}{r}u_\theta^1 + \frac{\partial u_z^1}{\partial z} = 0. \tag{26}$$

Given test functions $\boldsymbol{v}^1 \in V_0$ and $q^1 \in L_0^2$, and integrating by parts where appropriate, the weak forms of the linearized equations (23)–(26) are

$$
\begin{aligned}
R \int_\Omega & \left(-\gamma u_r^1 + u_r^0 \frac{\partial u_r^1}{\partial r} + u_r^1 \frac{\partial u_r^0}{\partial r} + u_z^0 \frac{\partial u_r^1}{\partial z} + u_z^1 \frac{\partial u_r^0}{\partial z} \right) v_r^1 \, dV \\
& - \int_\Omega p^1 \left(\frac{1}{r} \frac{\partial}{\partial r} (r v_r^1) \right) \, dV \\
& + \int_\Omega \left(\frac{\partial u_r^1}{\partial r} \frac{\partial v_r^1}{\partial r} + \frac{m^2}{r^2} u_r^1 v_r^1 + \frac{\partial u_r^1}{\partial z} \frac{\partial v_r^1}{\partial z} + \frac{1}{r^2} u_r^1 v_r^1 + \frac{2im}{r^2} u_\theta^1 v_r^1 \right) \, dV \\
& + \int_\Gamma \left(p^1 n_r - \frac{\partial u_r^1}{\partial r} n_r - \frac{\partial u_r^1}{\partial z} n_z \right) v_r^1 \, dS = 0,
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
R \int_\Omega & \left(-\gamma u_\theta^1 + u_r^0 \frac{\partial u_\theta^1}{\partial r} + u_z^0 \frac{\partial u_\theta^1}{\partial z} + \frac{u_r^0 u_\theta^1}{r} \right) v_\theta^1 \, dV \\
& + \int_\Omega \frac{im}{r} p^1 v_\theta^1 \, dV \\
& + \int_\Omega \left(\frac{\partial u_\theta^1}{\partial r} \frac{\partial v_\theta^1}{\partial r} + \frac{m^2}{r^2} u_\theta^1 v_\theta^1 + \frac{\partial u_\theta^1}{\partial z} \frac{\partial v_\theta^1}{\partial z} - \frac{2im}{r^2} u_r^1 v_\theta^1 + \frac{1}{r^2} u_\theta^1 v_\theta^1 \right) \, dV \\
& + \int_\Gamma \left(-\frac{\partial u_\theta^1}{\partial r} n_r - \frac{\partial u_\theta^1}{\partial z} n_z \right) v_\theta^1 \, dS = 0,
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
R \int_\Omega & \left(-\gamma u_z^1 + u_r^0 \frac{\partial u_z^1}{\partial r} + u_r^1 \frac{\partial u_z^0}{\partial r} + u_z^0 \frac{\partial u_z^1}{\partial z} + u_z^1 \frac{\partial u_z^0}{\partial z} \right) v_z^1 \, dV \\
& - \int_\Omega p^1 \frac{\partial v_z^1}{\partial z} \, dV \\
& + \int_\Omega \left(\frac{\partial u_z^1}{\partial r} \frac{\partial v_z^1}{\partial r} + \frac{m^2}{r^2} u_z^1 v_z^1 + \frac{\partial u_z^1}{\partial z} \frac{\partial v_z^1}{\partial z} \right) \, dV \\
& + \int_\Gamma \left(p^1 n_z - \frac{\partial u_z^1}{\partial r} n_r - \frac{\partial u_z^1}{\partial z} n_z \right) v_z^1 \, dS = 0,
\end{aligned} \tag{29}
$$

$$
\int_\Omega \left(\frac{1}{r} \frac{\partial}{\partial r} (r u_r^1) + \frac{im}{r} u_\theta^1 + \frac{\partial u_z^1}{\partial z} \right) q^1 \, dV = 0. \tag{30}
$$

Equations (27)–(30) may be transformed into a system of real equations by multiplying equation (28) by i and replacing iu_θ by u_θ in all equations wherever it appears.

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Figure captions

Fig. 1. Schematic representation of the computational domain.

Fig. 2. The shortest domain and coarsest grid on which computations were performed. Here the blockage ratio is 0.5. There are 300 biquadratic velocity, discontinuous linear pressure quadrilateral elements. The domain extends 3 pipe diameters upstream of the center of the sphere and 7 1/2 pipe diameters downstream.

Fig. 3. Contours of the velocity components of the null eigenvector for an azimuthal wavenumber of one and a blockage ratio of 0.5. (a) radial velocity component; (b) azimuthal velocity component; (c) axial velocity component.

Fig. 4. The most dangerous eigenvalue $\gamma = \mu + i\omega$ at Reynolds number $R_d = 350$ and blockage ratio $b = 0.5$, for azimuthal wavenumber m. (a) real part, μ; (b) imaginary part, ω.

Fig. 5. The critical Reynolds number R_d, as a function of blockage ratio b. For all blockage ratios in the range $0.2 \leq b \leq 0.7$, the loss of stability was associated with a real eigenvalue with azimuthal wavenumber 1.

Fig. 6. The critical Reynolds number R, as a function of blockage ratio b. For all blockage ratios in the range $0.2 \leq b \leq 0.7$ the loss of stability was associated with a real eigenvalue with azimuthal wavenumber 1.

Fig. 7. The length of the recirculation region as measured along the centerline from the rear of the sphere, and nondimensionalized with respect to the sphere diameter (l_d), and the pipe diameter (l_D). The length of the recirculation region at the critical Reynolds number is plotted as a function of blockage ratio b. $\triangle - .\triangle$, l_d; $\sqcup - .\sqcup$, l_D.

Fig. 8. The nondimensionalized pressure drop along the pipe wall from one pipe diameter upstream of the center of the sphere to one pipe diameter downstream of the center of the sphere. The pressure drop at the critical Reynolds number is plotted as a function of blockage ratio b. The value at $b = 0$ is the numerically and analytically determined pressure drop corresponding to Poiseuille flow over an equal length of unblocked pipe.