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OXFORD CENTRE FOR COLLABORATIVE APPLIED MATHEMATICS

Adaptive solution of coupled time-dependent systems

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The coupled parabolic problem

Find functions $u_i(x,t)$, $x \in \Omega_i$, $t \in [0,T]$, for $i = 1,2$, where Ω_i are non-overlapping open polygonal domains sharing a common interface Γ. Here, $u_1(x,t)$ and $u_2(x,t)$ satisfy the following partial differential equations, continuity constraints along Γ and initial and boundary values,

$$\frac{\partial u_i}{\partial t} + \mathcal{L}_i u_i = f_i, \quad \text{in } \Omega_i \times (0,T],\ i=1,2, \tag{1}$$
$$u_i(x,0) = u_{i,0}(x), \quad \text{for all } x \in \Omega_i,\ i=1,2, \tag{2}$$
$$u_i = u_i^D, \quad \text{on } \Gamma_i^D \times (0,T],\ i=1,2, \tag{3}$$
$$k_i \frac{\partial u_i}{\partial n_i} = 0, \quad \text{on } \Gamma_i^N \times (0,T],\ i=1,2, \tag{4}$$
$$u_1 = u_2, \quad \text{on } \Gamma \times (0,T], \tag{5}$$
$$k_1 \frac{\partial u_1}{\partial n_1} = -k_2 \frac{\partial u_2}{\partial n_2}, \quad \text{on } \Gamma \times (0,T], \tag{6}$$

where $\mathcal{L}_i u_i = -\nabla \cdot (k_i \nabla u_i) + c_i u_i$, $k_i \geq k_{i,0} > 0$, c_i and f_i are sufficiently smooth, and the domain boundaries $\partial\Omega_i = \Gamma_i^D \cup \Gamma_i^N \cup \Gamma$, $i = 1,2$, are such that $\Gamma_i^D \cap \Gamma_i^N = \emptyset$, $\Gamma_i^D \cap \Gamma = \emptyset$ and $\Gamma_i^N \cap \Gamma = \emptyset$.

Weak formulation

The weak formulation is: *Find* $(u_1, u_2, \lambda) \in X_{1,D} \times X_{2,D} \times Y$ *such that*

$$\int_0^T \left\{ \mp < \lambda, v_i > + \left(\frac{\partial u_i}{\partial t}, v_i \right) + a_i(u_i, v_i) - (f_i, v_i) \right\} \,\mathrm{d}t = 0 \quad \forall v_i \in X_{i,0},\ i=1,2, \tag{7}$$
$$\int_0^T < \nu, u_1 - u_2 > \,\mathrm{d}t = 0 \ \forall \nu \in Y, \tag{8}$$
$$u_i(x,0) = u_{i,0}(x),\ i=1,2, \tag{9}$$

where function spaces $X_i = L_2(0,T; H^1(\Omega_i))$, $i = 1,2$, and $Y = L_2(0,T; H^{-1/2}(\Gamma))$, and

$$X_{i,D} = \{ v \in X_i \mid v = u_i^D \ \forall (x,t) \in \Gamma_i^D \times I \},\ i=1,2,$$
$$X_{i,0} = \{ v \in X_i \mid v = 0 \ \forall (x,t) \in \Gamma_i^D \times I \},\ i=1,2,$$

are the affine subspaces including the given Dirichlet and zero Dirichlet boundary conditions respectively along Γ_i^D. Here

$$(v_i, w_i) = \int_{\Omega_i} v_i w_i \,\mathrm{d}x,$$
$$a_i(v_i, w_i) = \int_{\Omega_i} (k_i \nabla v_i \cdot \nabla w_i + c_i v_i w_i) \,\mathrm{d}x$$

are defined $\forall v_i, w_i \in X_i$, $i = 1,2$. Finally, for any pair $(v, \nu) \in X_i \times Y$ we define $< \nu, v >$ to be the standard duality pairing.

Quantity of Interest

Let the discretization errors be defined as $e_i = u_i - u_i^h$, $i = 1,2$ and $e_\lambda = \lambda - \lambda^h$, which are functions of both space and time. The quantity of interest

$$(QoI) = \int_0^T \{ - < \lambda, \psi > + (u_1, \psi_1) + (u_2, \psi_2) \} \,\mathrm{d}t,$$

is a linear functional of the solution with coefficients ψ_1, ψ_2 and ψ, and has error

$$\mathcal{E} = \int_0^T \{ - < e_\lambda, \psi > + (e_1, \psi_1) + (e_2, \psi_2) \} \,\mathrm{d}t. \tag{10}$$

Error estimator

The adjoint problem corresponding to (7)–(9) is: *Find* $(\phi_1, \phi_2, \mu) \in X_{1,0} \times X_{2,0} \times Y$ *such that*

$$\int_0^T \left\{ \mp < \mu, v_i > - \left(\frac{\partial \phi_i}{\partial t}, v_i \right) + a_i(\phi_i, v_i) - (\psi_i, v_i) \right\} \,\mathrm{d}t = 0 \quad \forall v_i \in X_{i,0},\ i=1,2, \tag{11}$$
$$\int_0^T < \rho, \phi_1 - \phi_2 > \,\mathrm{d}t = \int_0^T < \rho, \psi > \,\mathrm{d}t \quad \forall \rho \in Y, \tag{12}$$
$$\phi_i(x,T) = 0,\ i=1,2. \tag{13}$$

The error $\mathcal{E}$ defined by (10) can be computed as

$$\mathcal{E} = \sum_{i=1}^{2} \left\{ (\phi_{i,0}, u_{i,0} - \pi u_{i,0}) - \sum_{n=1}^{N} (\phi_{i,n-1}, [u_i^h]_{n-1}) + \sum_{n=1}^{N} \int_{I_n} \left\{ \pm < \mu, u_i^h > + (R_i, \phi_i) \right\} \mathrm{d}t \right\},$$

where (ϕ_1, ϕ_2, μ) is the solution of the adjoint problem (11)-(13) and $R_i, i = 1,2$ are appropriately defined generalized residuals.

Solving both forward and adjoint problems via suitable finite element methods, a *computable* error estimate is

$$\begin{aligned}
\mathcal{E}_{est} = \sum_{i=1}^{2} \Big\{ & (\phi_{i,0+}^h, u_{i,0} - \pi^h u_{i,0}) && \text{initial error} \\
& - \sum_{n=1}^{N} \left(\phi_{i,(n-1)+}^h - \theta\phi_{i,(n-1)+}^h, \left[u_i^h\right]_{n-1} \right) && \text{temporal error} \\
& + \sum_{n=1}^{N} \Big\{ \int_{I_n} \pm < \mu^h - \theta\mu^h, u_i^h > \,\mathrm{d}t && \text{temporal error} \\
& \quad + (R_i, \phi_i^h - \theta\phi_i^h) \Big\} \,\mathrm{d}t && \text{temporal error} \\
& - \sum_{n=1}^{N} \left(\theta\phi_{i,(n-1)+}^h - \pi^h\theta\phi_{i,(n-1)+}^h, \left[u_i^h\right]_{n-1} \right) && \text{spatial error} \\
& + \sum_{n=1}^{N} \int_{I_n} \Big\{ \pm < \theta\mu^h - \pi^h\theta\mu^h, u_i^h > && \text{spatial error} \\
& \quad + (R_i, \theta\phi_i^h - \pi^h\theta\phi_i^h) \Big\} \,\mathrm{d}t \Big\}, && \text{spatial error}
\end{aligned}$$

where θ and π_i^h, $i = 1,2$ are temporal and spatial projections respectively.

We write the total error as the sum of spatial errors (including initial errors) $\mathcal{E}_x$, and temporal errors $\mathcal{E}_t$, and set $\mathcal{E}_{est} = \mathcal{E}_x + \mathcal{E}_t$. We sum the absolute values of element contributions to define $\overline{\mathcal{E}}_{est} = \overline{\mathcal{E}}_x + \overline{\mathcal{E}}_t$ and effectivity ratios

$$\mathcal{R} = \frac{\mathcal{E}_{est}}{\mathcal{E}}, \quad \text{and} \quad \overline{\mathcal{R}} = \frac{\overline{\mathcal{E}}_{est}}{|\mathcal{E}|}.$$

Effectivity of the error estimator

Let $\Omega_1 = (0,1) \times (0,1)$ and $\Omega_2 = (0,1) \times (1,2)$. We solve the following test problems on meshes which both conform and fail to conform across the interface.

(i) $\mathcal{L}_1 u = \mathcal{L}_2 u = -\Delta u$, with exact solutions $u_1 = 0.5ty(2-y)$ and $u_2 = -0.5t\cos(0.5\pi(3-y))$.

(ii) $\mathcal{L}_1 u = -\Delta u$ and $\mathcal{L}_2 u = -0.1\Delta u + u$, with exact solutions $u_1 = t\sin(\pi y)$ and $u_2 = 10t\sin(\pi y)$.

(iii) Elliptic-parabolic coupling, $-\Delta u_1 = f_1$, $\frac{\partial u_2}{\partial t} - \Delta u_2 = f_2$, with exact solutions $u_1 = u_2 = t\sin(\pi y)$.

	(i)		(ii)		(iii)	
r	conf	non-conf	conf	non-conf	conf	non-conf
	$\mathcal{R}$					
1	1.0140	1.0091	1.0056	0.9974	1.0253	1.0137
2	1.0035	1.0023	1.0023	0.9993	1.0063	1.0034
3	1.0009	1.0006	1.0015	0.9997	1.0016	1.0008
	$\overline{\mathcal{R}}$					
1	1.2450	1.2035	1.3760	1.3416	1.4542	1.3654
2	1.2914	1.2429	1.3974	1.3609	1.5416	1.4388
3	1.3029	1.2528	1.4027	1.3657	1.5634	1.4571

Table 1: Effectivity ratios $\mathcal{R}$ and $\overline{\mathcal{R}}$, for model problems (i)-(iii) varying the spatial accuracy of the adjoint solution. The value of r corresponds to the number of uniform spatial refinements used to obtain the adjoint mesh from the original mesh.

Spatial adaptivity for elliptic problems

Let $\mathcal{L}_1 u = \mathcal{L}_2 u = -\Delta u$ and chose f such that $u_1 = u_2 = \sin(\pi x)\sin(\pi y)$ using quadratic finite elements in Ω_1 and linear finite elements in Ω_2

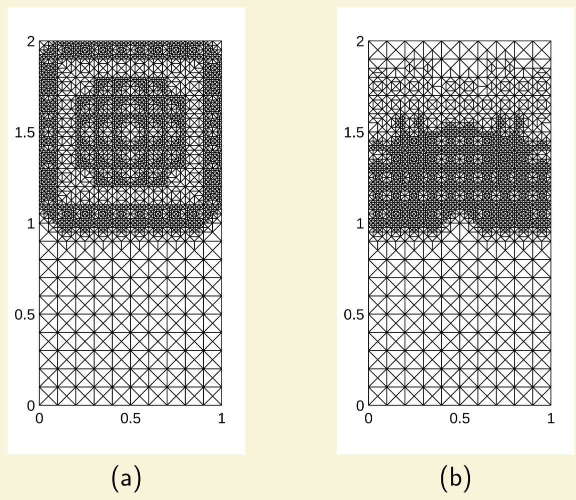

Figure 1: (a) Refinement based on error in the domain. (b) Refinement based on error on the interface.

Refinement strategies based on either (QoI) result in significantly more linear elements in Ω_2 than quadratic elements in Ω_1.

Space-time block-adaptivity for parabolic problems

Refinement based on domain error

Let $\mathcal{L}_1 u = \mathcal{L}_2 u = -\Delta u$ and choose right-hand sides, the initial and the Dirichlet boundary conditions so that

$$u = \exp(-100(x - 0.5 - 0.25\sin(\pi t))^2 - 100(y - 1 - 0.5\cos(\pi t))^2).$$

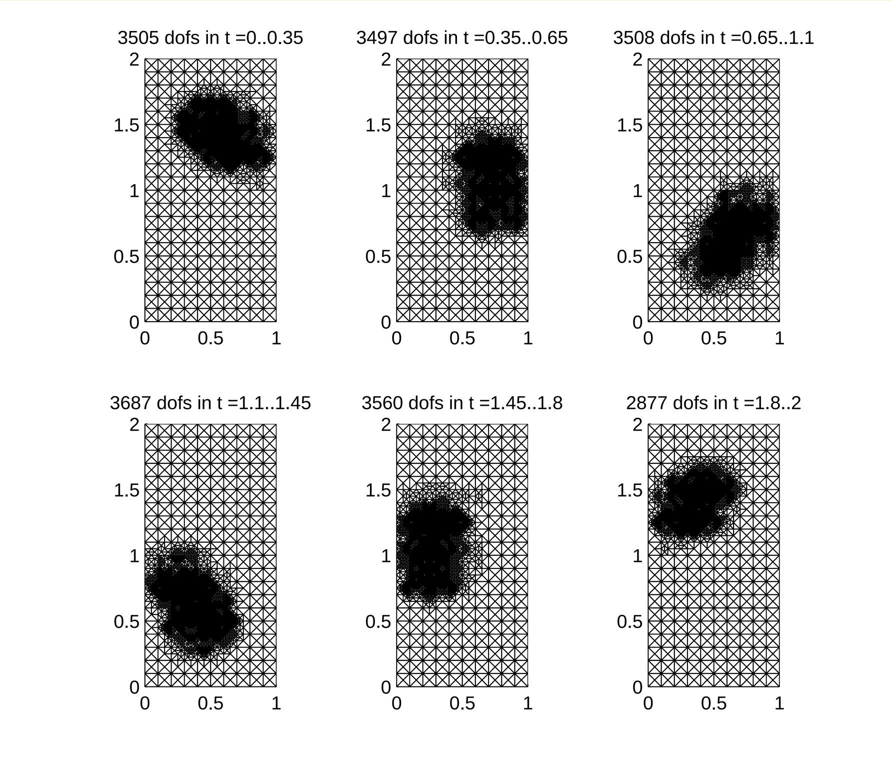

Figure 2: Space/time blocks obtained by a block-adaptive refinement strategy.

	$10^4\mathcal{E}_{est}$	$10^4\mathcal{E}_t$	$10^4\mathcal{E}_x$	e
original	-3.3234	-0.2335	-3.0899	0.0895
refined	-0.2857	-0.0545	-0.2311	0.0621

The block-refinement strategy results in six space-time blocks with spatial refinement clustered around the location of the Gaussian "bump". Refinement leads to a significant reduction in the QoI but a very much smaller reduction in the L_2-error, e.

Refinement based on interface error

Let $\mathcal{L}_1 u = -0.2\Delta u$ and $\mathcal{L}_2 u = -\Delta u$.

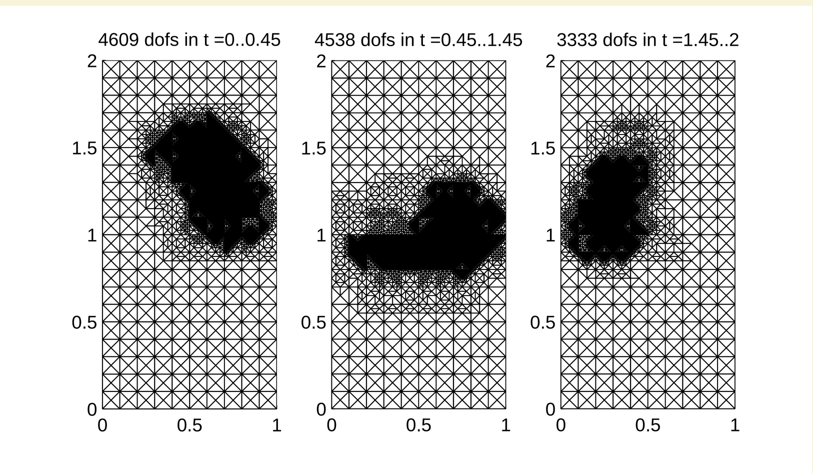

Figure 3: Space/time blocks obtained by a block-adaptive refinement strategy.

Interface	$10^4\mathcal{E}_{est}$	$10^4\mathcal{E}_t$	$10^4\mathcal{E}_x$	e
original	-7.8804	-5.6943	-2.1862	0.0153
refined	-1.5004	-1.5120	0.0116	0.0017

The block-refinement strategy results in three space-time blocks. Less spatial refinement is required after the "bump" has (re)-crossed the interface and does not return. Refinement leads to a significant reduction in both the QoI and in the L_2-error (in the Lagrange multiplier) e.

Conclusions and future directions

- An adjoint-based *a posteriori* analysis provides accurate error estimates for quantities of interest that may include both the solutions and the normal flux across the interface. A blockwise refinement strategy based on these estimates reduces the error in the quantity of interest and the usual L_2 error to a lesser extent.
- Extensions currently under way include Stokes-Darcy coupling and fluid-solid interaction problems.