Stability of flow past a confined cylinder

Andrew Cliffe[†] and Simon Tavener[‡]

University of Nottingham[†] and Colorado State University[‡]

Flow past a cylinder

The two-dimensional flow of an incompressible Newtonian fluid past a cylinder in a laterally “unbounded” domain has been shown to lose stability at a supercritical Hopf bifurcation point by Jackson (1987).

Flow past a cylinder

The two-dimensional flow of an incompressible Newtonian fluid past a cylinder in a laterally “unbounded” domain has been shown to lose stability at a supercritical Hopf bifurcation point by Jackson (1987).

Chen, Pritchard & T. (1995) later showed a similar result when the flow is confined in a two-dimensional channel.

Flow past a cylinder

The two-dimensional flow of an incompressible Newtonian fluid past a cylinder in a laterally "unbounded" domain has been shown to lose stability at a supercritical Hopf bifurcation point by Jackson (1987).

Chen, Pritchard & T. (1995) later showed a similar result when the flow is confined in a two-dimensional channel.

Rotation of the cylinder about its axis is known to delay the onset of a periodic wake and to completely eliminate the vortex street above a sufficiently large rotation rate (Jaminet & van Atta, 1969).

Flow past a cylinder

The two-dimensional flow of an incompressible Newtonian fluid past a cylinder in a laterally “unbounded” domain has been shown to lose stability at a supercritical Hopf bifurcation point by Jackson (1987).

Chen, Pritchard & T. (1995) later showed a similar result when the flow is confined in a two-dimensional channel.

Rotation of the cylinder about its axis is known to delay the onset of a periodic wake and to completely eliminate the vortex street above a sufficiently large rotation rate (Jaminet & van Atta, 1969).

In laterally “unbounded” domains the vortex street is suppressed when the tangential velocity at the surface of the cylinder is approximately twice the (uniform) inlet velocity.

Flow past a cylinder

The two-dimensional flow of an incompressible Newtonian fluid past a cylinder in a laterally “unbounded” domain has been shown to lose stability at a supercritical Hopf bifurcation point by Jackson (1987).

Chen, Pritchard & T. (1995) later showed a similar result when the flow is confined in a two-dimensional channel.

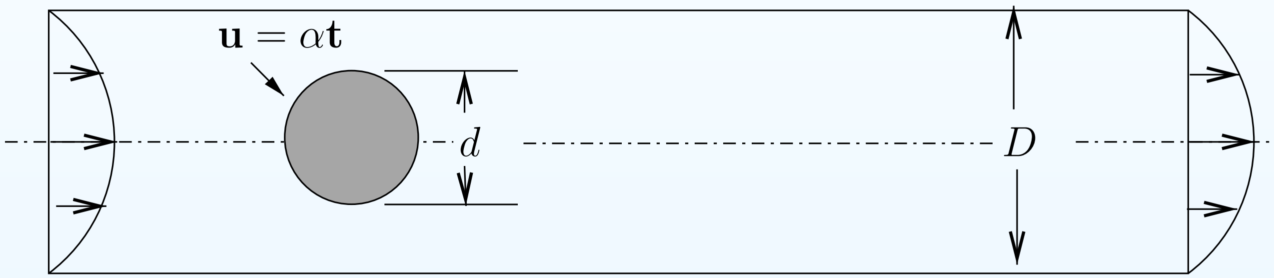
Rotation of the cylinder about its axis is known to delay the onset of a periodic wake and to completely eliminate the vortex street above a sufficiently large rotation rate (Jaminet & van Atta, 1969).

In laterally “unbounded” domains the vortex street is suppressed when the tangential velocity at the surface of the cylinder is approximately twice the (uniform) inlet velocity.

If the instability occurs at a Hopf bifurcation point, *what happens to this singularity as the cylinder rotation is increased?*

Problem definition

Consider the two-dimensional flow of an incompressible, Newtonian fluid in the domain shown below.



$$\left.\begin{aligned} R(\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) + \nabla p - \nabla^2 \boldsymbol{u} &= 0 \\ \nabla\cdot\boldsymbol{u} &= 0 \end{aligned}\right\} \text{ in } \Omega, \quad \text{with} \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on} \quad \partial\Omega.$$

where we define the Reynolds number, R as

$$R = \frac{d\,U_d}{\nu} \quad \text{where} \quad U_d = \frac{1}{d}\int_{-d/2}^{d/2} u_{\text{inlet}}(y)\;\mathrm{d}y\;.$$

Non-dimensional parameters

We nondimensionalize the rotation rate of the cylinder by defining

$$\alpha = \frac{v_{\mathrm{tang}}}{U_d} ,$$

where v_{tang} is the tangential velocity at the surface of the cylinder.

The Strouhal number of the periodic flow is

$$S = \frac{d\,f}{U_d} .$$

We define the blockage ratio B to be

$$B = \frac{d}{D} .$$

Mixed Finite Element Discretization

Define the finite dimensional subspaces $X_h \subset H_0^1(\Omega)$ and $M_h \subset L_0^2(\Omega)$.

Mixed Finite Element Discretization

Define the finite dimensional subspaces $X_h \subset H_0^1(\Omega)$ and $M_h \subset L_0^2(\Omega)$.

Let $\boldsymbol{u}_0 \in H^1(\Omega)$ be such that $\boldsymbol{u}_0 = \boldsymbol{g}$ on $\partial\Omega$.

Mixed Finite Element Discretization

Define the finite dimensional subspaces $X_h \subset H_0^1(\Omega)$ and $M_h \subset L_0^2(\Omega)$.

Let $\boldsymbol{u}_0 \in H^1(\Omega)$ be such that $\boldsymbol{u}_0 = \boldsymbol{g}$ on $\partial\Omega$.

Find $(\boldsymbol{u}_h - \boldsymbol{u}_0, p_h) \in X_h \times M_h$ such that

$$\begin{aligned}\int_\Omega \left[R\left(\frac{\partial \boldsymbol{u}_h}{\partial t} + \boldsymbol{u}_h \cdot \nabla \boldsymbol{u}_h\right)\boldsymbol{v}_h - p_h \nabla \cdot \boldsymbol{v}_h + \nabla \boldsymbol{u}_h : \nabla \boldsymbol{v}_h \right] &= 0 \quad \forall \boldsymbol{v}_h \in X_h, \\ -\int_\Omega q_h \nabla \cdot \boldsymbol{u}_h &= 0 \quad \forall q_h \in M_h.\end{aligned}$$

Mixed Finite Element Discretization

Define the finite dimensional subspaces $X_h \subset H_0^1(\Omega)$ and $M_h \subset L_0^2(\Omega)$.

Let $\boldsymbol{u}_0 \in H^1(\Omega)$ be such that $\boldsymbol{u}_0 = \boldsymbol{g}$ on $\partial\Omega$.

Find $(\boldsymbol{u}_h - \boldsymbol{u}_0, p_h) \in X_h \times M_h$ such that

$$\begin{aligned} \int_\Omega \left[R \left(\frac{\partial \boldsymbol{u}_h}{\partial t} + \boldsymbol{u}_h \cdot \nabla \boldsymbol{u}_h \right) \boldsymbol{v}_h - p_h \nabla \cdot \boldsymbol{v}_h + \nabla \boldsymbol{u}_h : \nabla \boldsymbol{v}_h \right] &= 0 \quad \forall \boldsymbol{v}_h \in X_h, \\ - \int_\Omega q_h \nabla \cdot \boldsymbol{u}_h &= 0 \quad \forall q_h \in M_h. \end{aligned}$$

We construct X_h using biquadratic quadrilateral elements and M_h using discontinuous piecewise linear approximation on these elements.

A priori approximation results

If $(\boldsymbol{u}, p)$ is a regular solution of the N.-S. equations then there is a solution of the discrete equations $(\boldsymbol{u}_h, p_h)$ such that

$$||\boldsymbol{u}_h - \boldsymbol{u}|| + ||p_h - p||_0 < Ch^2.$$

A priori approximation results

If $(\boldsymbol{u}, p)$ is a regular solution of the N.-S. equations then there is a solution of the discrete equations $(\boldsymbol{u}_h, p_h)$ such that

$$||\boldsymbol{u}_h - \boldsymbol{u}|| + ||p_h - p||_0 < Ch^2.$$

If $(\boldsymbol{u}, p, R)$ is a simple bifurcation point of the N.-S. equations, then there is a simple bifurcation point of the discrete equations $(\boldsymbol{u}_h, p_h, R_h)$ such that

$$\begin{aligned} ||\boldsymbol{u}_h - \boldsymbol{u}|| + ||p_h - p||_0 + |R_h - R| &< Ch^2, \\ |R_h - R| &< Ch^4. \end{aligned}$$

Note: Super-convergence in the bifurcation parameter.

Hopf bifurcation theorem

Consider a system of differential equations of the form

$$\frac{d\boldsymbol{x}}{dt} + \boldsymbol{f}(\boldsymbol{x}, \lambda) = \mathbf{0}; \quad \boldsymbol{x} \in \mathbb{R}^N,\ \lambda \in \mathbb{R}. \tag{1}$$

(We can write the discrete N.-S. equations in such form.)

If $\boldsymbol{f}(\mathbf{0}, 0) = \mathbf{0}$ is a steady solution of (1) and

(1) $\boldsymbol{f}_{\boldsymbol{x}}(\mathbf{0}, 0)$ has simple eigenvalues $\pm i\omega$,

(2) $\boldsymbol{f}_{\boldsymbol{x}}(\mathbf{0}, 0)$ has no other imaginary eigenvalues,

(3) $\sigma'(0) \neq 0$ where $\sigma(\lambda)$ is the real part of the imaginary eigenvalue (pair) of $\boldsymbol{f}_{\boldsymbol{x}}(\mathbf{0}, \lambda)$,

then there is a one-parameter family of periodic orbits of (1), $\boldsymbol{x}(t, \lambda)$ bifurcating from the steady solution at $\boldsymbol{x} = \mathbf{0}$ at $\lambda = 0$.

Griewank and Reddien

Consider the "extended system"

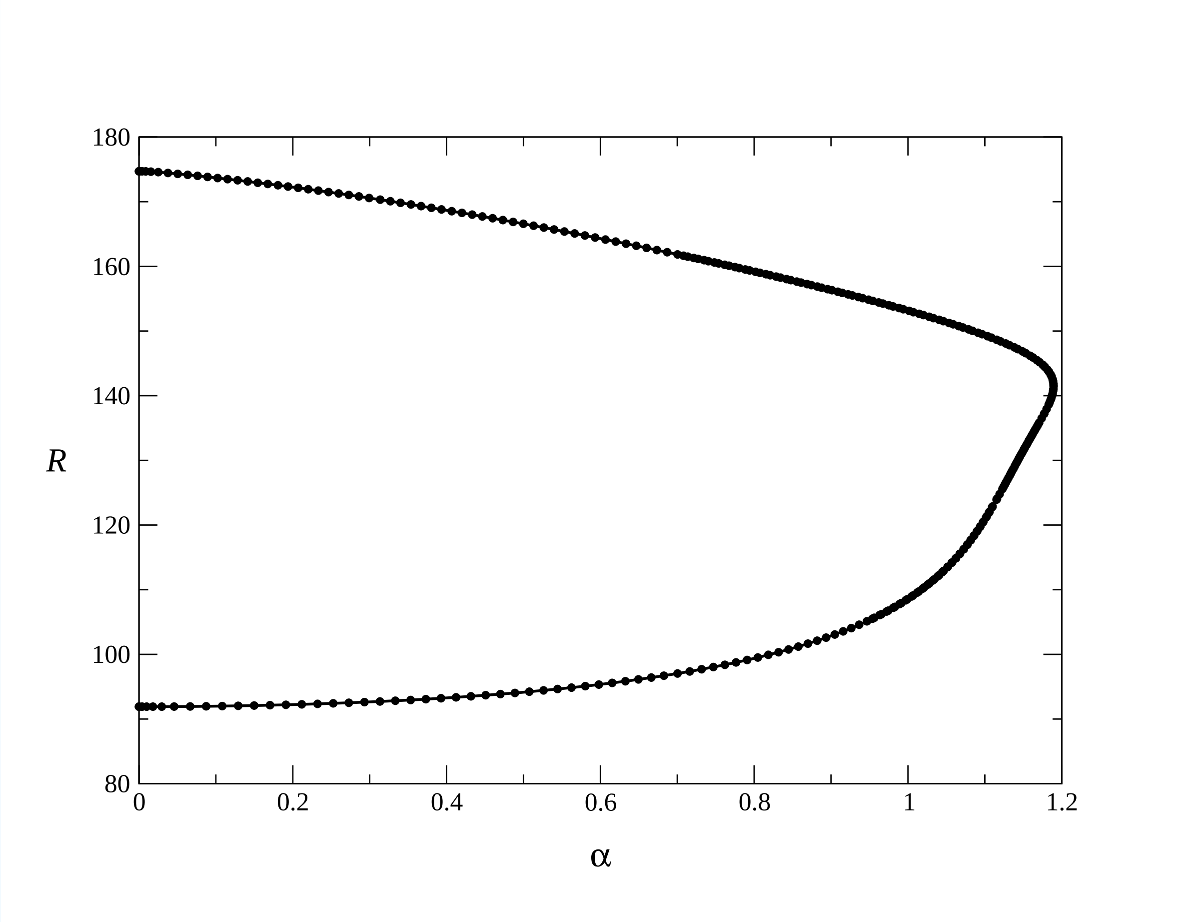
$$\begin{aligned} \boldsymbol{f} &= \boldsymbol{0} \\ \boldsymbol{f_x v} + i\omega \boldsymbol{v} &= \boldsymbol{0} \\ l(\boldsymbol{v}) &= 1 \end{aligned} \tag{2}$$

If $(\boldsymbol{x}, \lambda)$ is a Hopf point, then (2) is regular at $(\boldsymbol{x}, \lambda, \omega, \boldsymbol{v})$ and can be solved by Newton iteration.

The terms required for the computation of the Jacobian of (2) were determined using a computer algebra system.

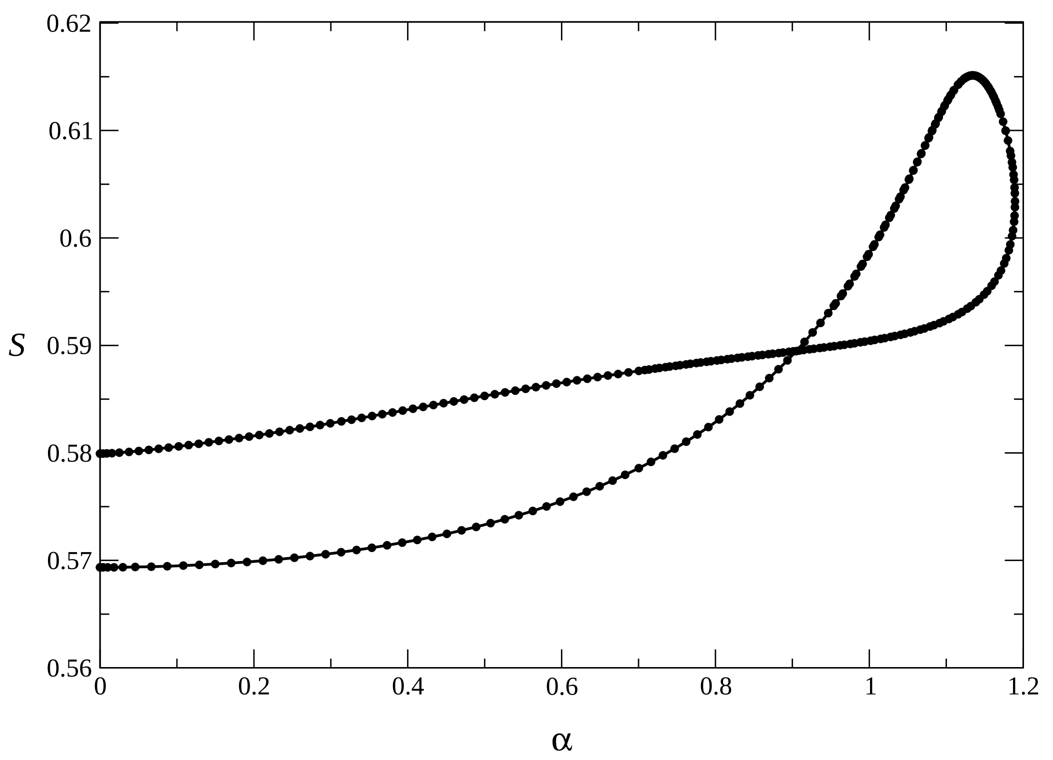
Initial estimates of the location of the Hopf bifurcation points were determined by finding the "most dangerous" eigenvalues of the linearized stability problem using the transform technique of Cliffe, Garratt & Spence (1993).

B=0.7: Hopf bifurcation



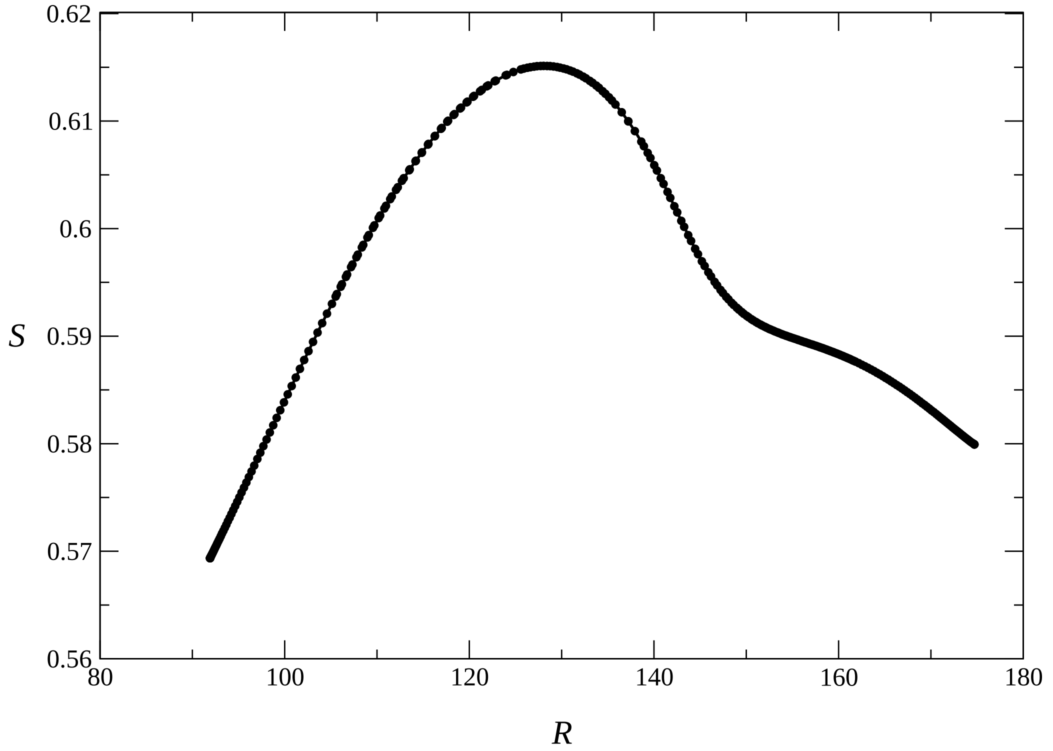
Reynolds number at the Hopf bifurcation point vs. rotation rate α.

B=0.7: Hopf bifurcation



Strouhal number at the Hopf bifurcation point vs. rotation rate α.

B=0.7: Hopf bifurcation



Strouhal number at the Hopf bifurcation point vs. Reynolds number.

B=0.7: Convergence study

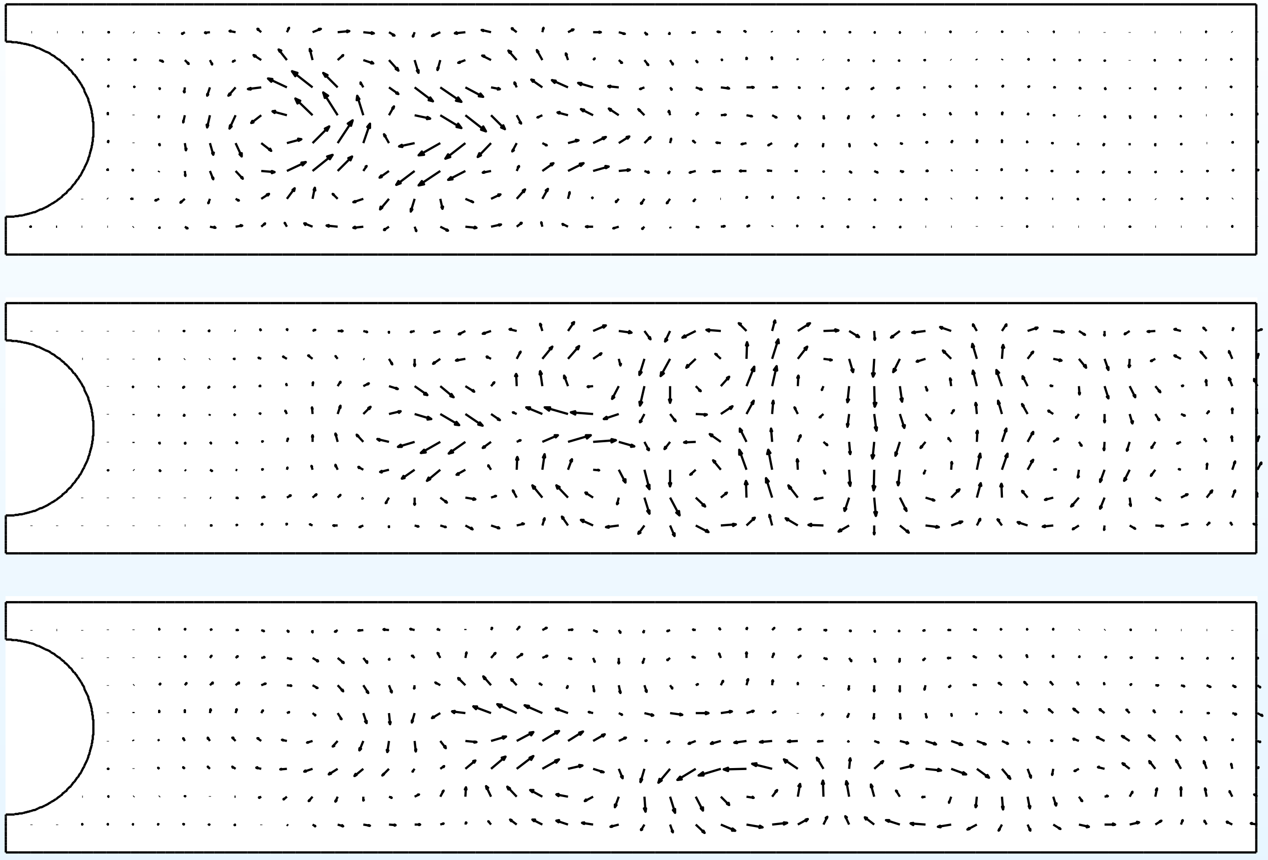
# elements	R	S	$\Delta R/h^4$	$\Delta R/h^2$
900	93.6882	0.5676		
3600	91.9125	0.5694	28.4	7.1
8100	91.9811	0.5694	5.56	0.62
14400	91.9944	0.5694	2.84	0.18
900	154.1212	0.5724		
3600	174.7059	0.5799		
8100	174.7365	0.5806		
14400	174.7156	0.5806		

B=0.7: Null eigenvectors - velocity field

(a)

(b)

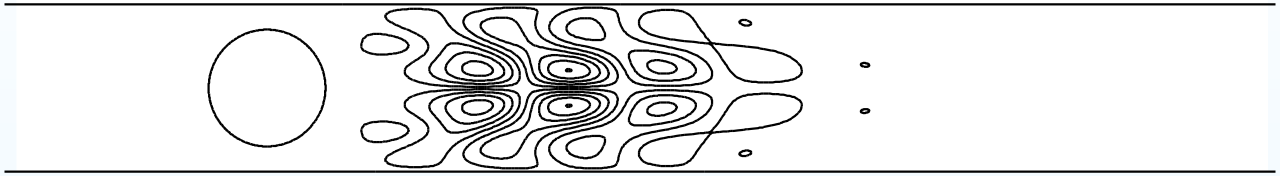
(c)



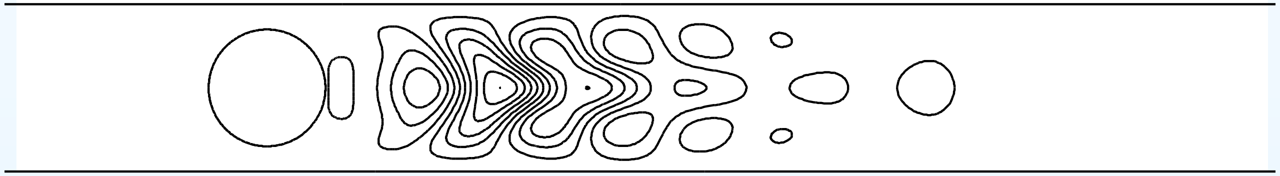
(a) $\alpha = 0, R = 92.0$, (b) $\alpha = 0, R = 174.7$; (c) $\alpha = 1.189, R = 140.7$.

B=0.7: Eigenvector velocity components

(a)



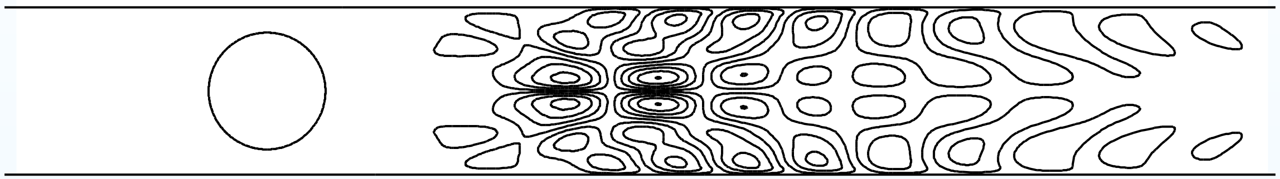
(b)



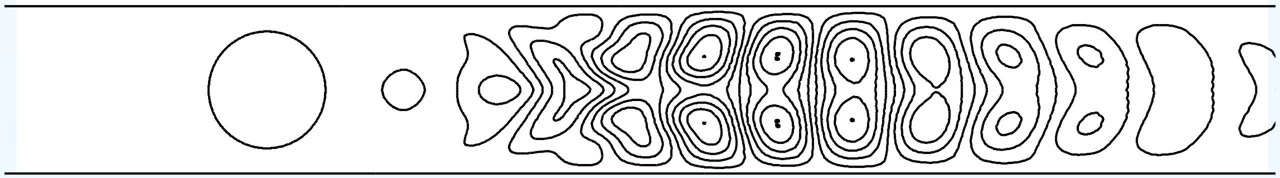
Velocity components of null-eigenvector at the supercritical Hopf bifurcation point at $\alpha = 0, R = 92.0$. The contours are equally spaced.

B=0.7: Eigenvector velocity components

(a)



(b)

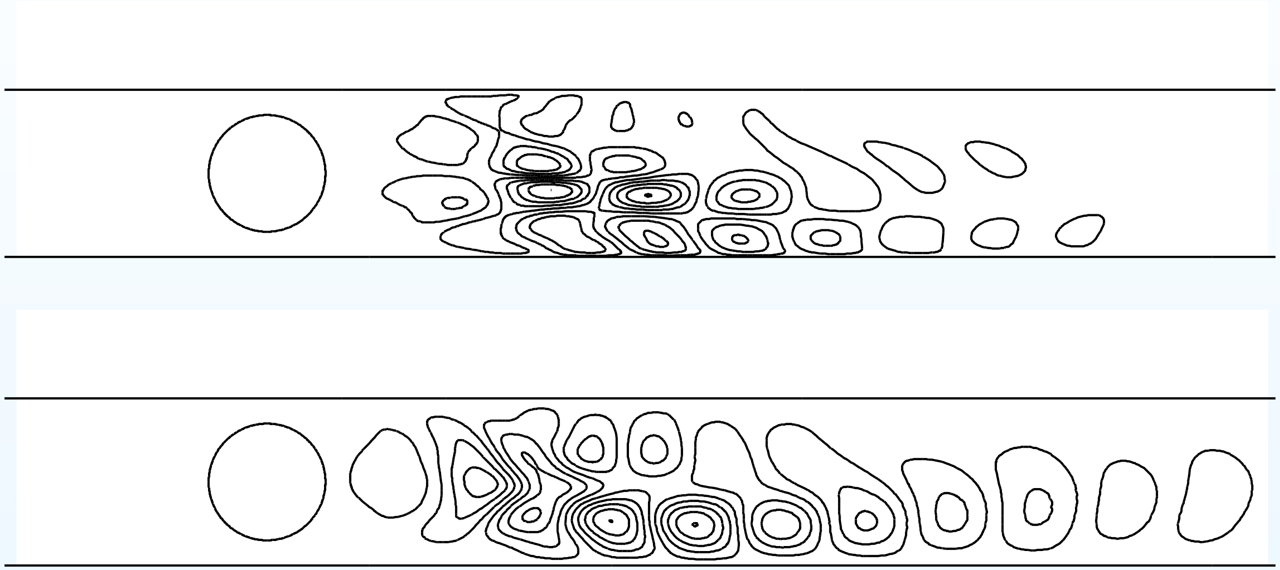


Velocity components of null-eigenvector at the supercritical Hopf bifurcation point at $\alpha = 0, R = 174.7$. The contours are equally spaced.

B=0.7: Eigenvector velocity components

(a)

(b)



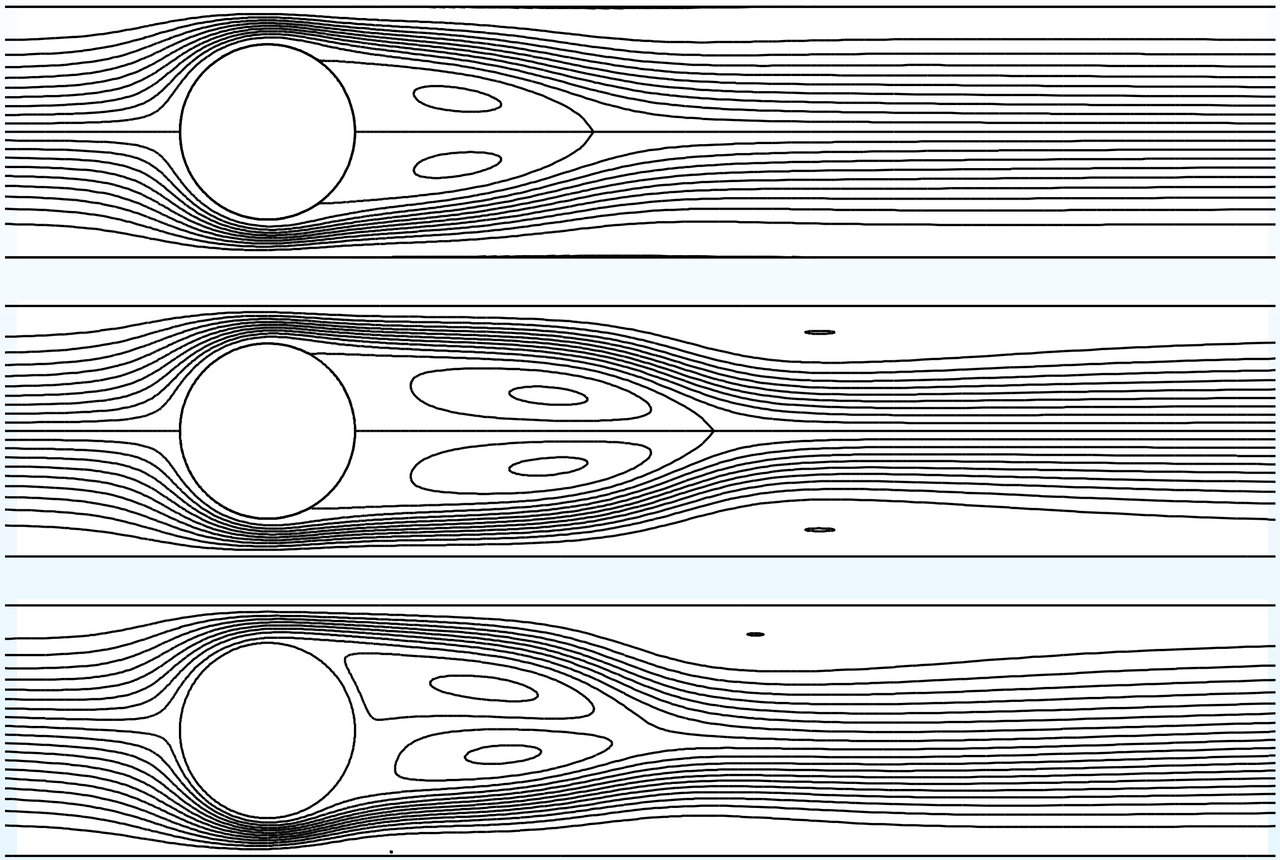
Velocity components of null-eigenvector at the supercritical Hopf bifurcation point at $\alpha = 1.189, R = 140.7$. The contours are equally spaced.

B=0.7: Streamlines of “critical” flows

(a)

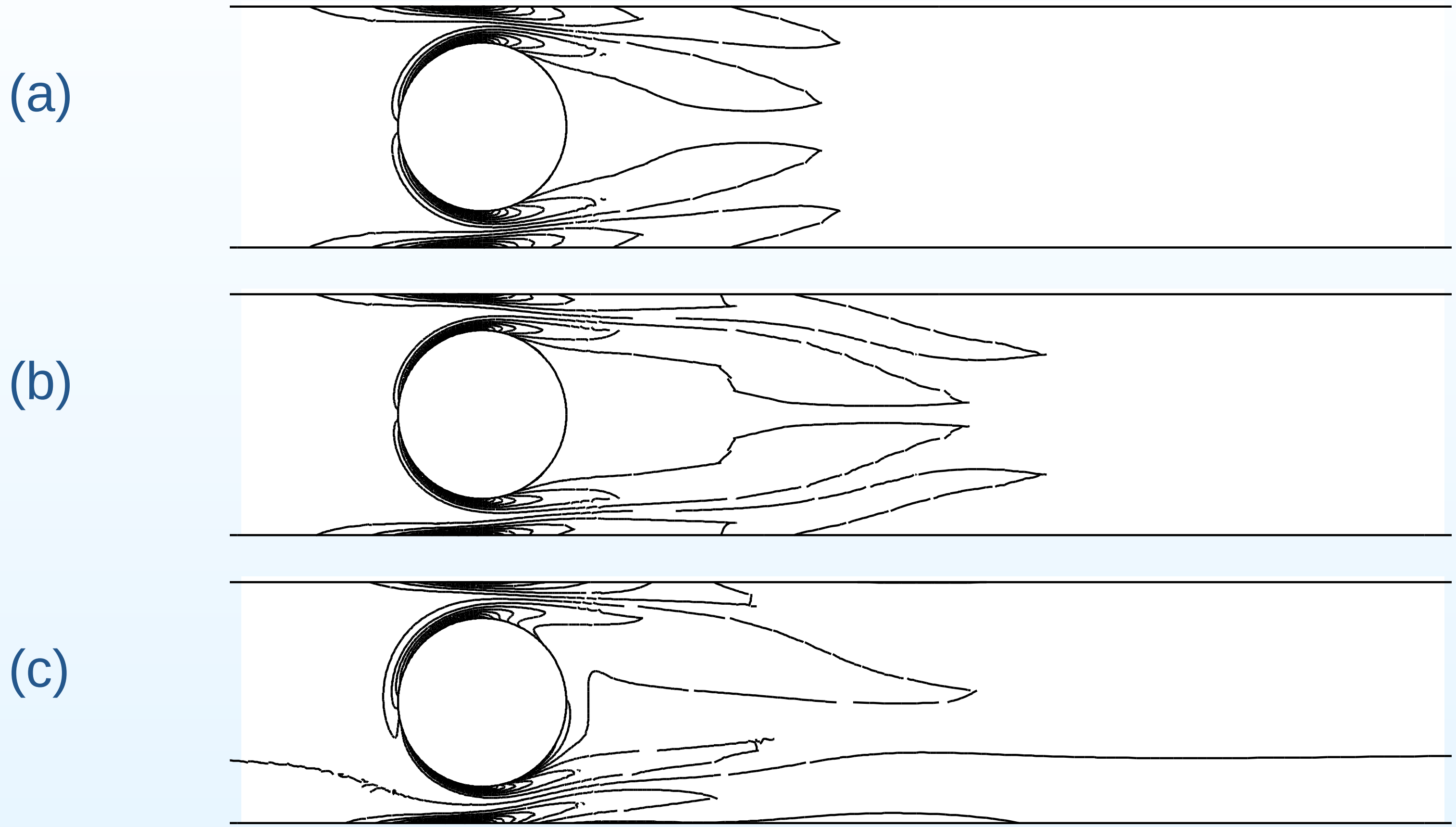
(b)

(c)



(a) $\alpha = 0, R = 92.0$, (b) $\alpha = 0, R = 174.7$; (c) $\alpha = 1.189, R = 140.7$. The contours are equally spaced.

B=0.7: Vorticity contours of “critical” flows



(a) $\alpha = 0, R = 92.0$, (b) $\alpha = 0, R = 174.7$; (c) $\alpha = 1.189, R = 140.7$. The contours are equally spaced.

B=0.7: Time-dependent calculations

Time-dependent simulations were performed using an implementation of the variable time-step method of Byrne and Hindmarsh (1975) based on backward difference formulas of orders 1–5.

B=0.7: Time-dependent calculations

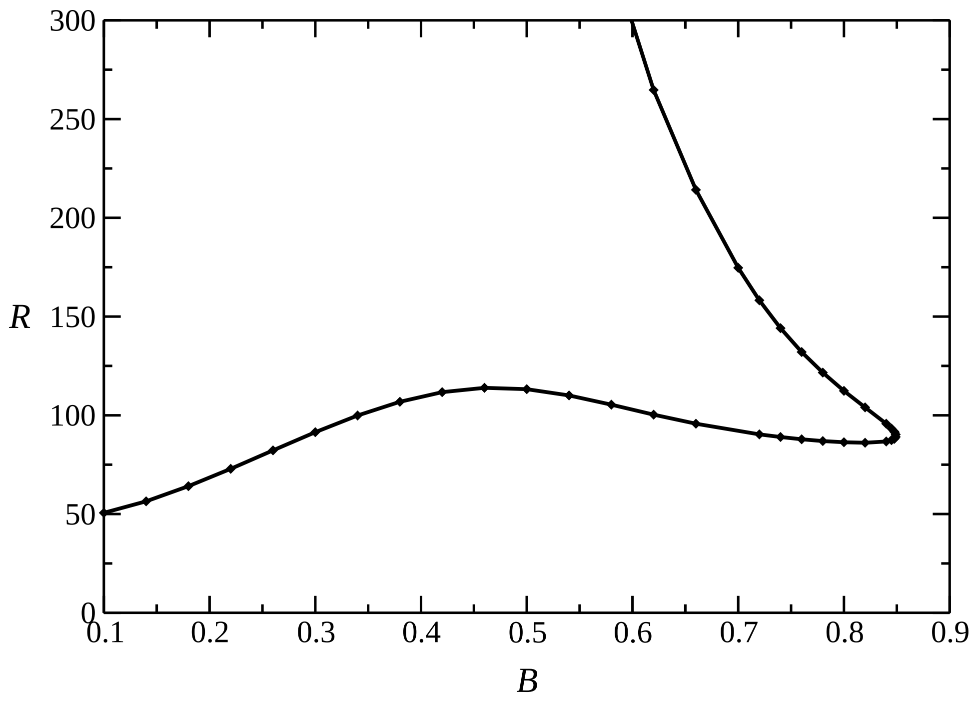
R	α	solution
105.4	0	periodic
105.4	0.008	periodic
105.4	0.080	periodic
105.4	0.797	periodic
105.4	0.877	???
105.4	0.956	steady
105.4	1.036	steady
105.4	1.115	steady
105.4	1.119	steady
105.4	0	periodic
140.6	0	periodic
158.1	0	steady
175.7	0	steady

Other blockage ratios

At $B = 0.7$ there two Hopf bifurcations (supercritical and subcritical) at $\alpha = 0$, i.e., for a non-rotating cylinder. *How did we miss this before?*

Other blockage ratios

At $B = 0.7$ there two Hopf bifurcations (supercritical and subcritical) at $\alpha = 0$, i.e., for a non-rotating cylinder. *How did we miss this before?*



Locus of Hopf bifurcation points as a function blockage ratio for a stationary cylinder.

Computation of periodic orbits

A solution $\boldsymbol{x}(t)$ is periodic if there exists a $\tau > 0$ such that $\boldsymbol{x}(t+\tau) = \boldsymbol{x}(t)$ for all $t \geq 0$. The minimal such τ is the period T.

We define the *flow* $\boldsymbol{\phi}(\boldsymbol{x}(0), t)$ as the solution at time t with initial condition $\boldsymbol{x}(0)$. One method to compute periodic orbits is to solve

$$\begin{pmatrix} \boldsymbol{\phi}(\boldsymbol{x}, T) - \boldsymbol{x} \\ \psi(\boldsymbol{x}, T) \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ 0 \end{pmatrix}$$

via Newton's method (or Newton-Picard iteration) for $\boldsymbol{x}^*(0)$ and period T^*, i.e., to solve

$$\begin{bmatrix} M^{(k)} - I & \phi_T^{(k)} \\ \psi_{\boldsymbol{x}}^{(k)} & \psi_T^{(k)} \end{bmatrix} \begin{pmatrix} \Delta \boldsymbol{x}^{(k)} \\ \Delta T^{(k)} \end{pmatrix} = - \begin{pmatrix} \boldsymbol{r}(\boldsymbol{x}^{(k)}, T^{(k)}) \\ \psi(\boldsymbol{x}^{(k)}, T^{(k)}) \end{pmatrix}$$

where $\psi(\boldsymbol{x}, T)$ is a phase condition, M is the Jacobian matrix $\boldsymbol{\phi}_{\boldsymbol{x}}$ and $\boldsymbol{r}(\boldsymbol{x}, T) = \boldsymbol{\phi}(\boldsymbol{x}, T) - \boldsymbol{x}$.

Computation of periodic orbits

The stability of the periodic orbit $\boldsymbol{\phi}^*(t)$ is determined by the eigenvalues of the *Monodromy matrix*

$$M^* = \left.\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}}\right|_{(\boldsymbol{x}^*(0),T^*)},$$

which are known as the *Floquet multipliers*. The orbit $\boldsymbol{\phi}^*(t)$ is stable if all the Floquet multipliers lie within the unit circle.

Note: The Monodromy matrix $\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}}$ is part of the Jacobian of the system of equations.

Temporal symmetry

Consider the nonlinear equation

$$\boldsymbol{f}(\boldsymbol{x};\lambda) = \boldsymbol{0}; \quad \boldsymbol{f} : \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{R}^N.$$

where N is large.

Temporal symmetry

Consider the nonlinear equation

$$\boldsymbol{f}(\boldsymbol{x};\lambda) = \boldsymbol{0}; \quad \boldsymbol{f} : \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{R}^N.$$

where N is large.

A non-singular $(N \times N)$ matrix γ is a symmetry of these equations if

$$\boldsymbol{f}(\gamma\boldsymbol{x};\lambda) = \gamma\boldsymbol{f}(\boldsymbol{x};\lambda).$$

The set of all symmetries forms a group Γ.

Temporal symmetry

Let $C_{2\pi}$ be the space of continuous 2π periodic functions $\boldsymbol{x}(t)$, $\boldsymbol{x}(t) : \mathbb{R} \mapsto \mathbb{R}^N$.

Temporal symmetry

Let $C_{2\pi}$ be the space of continuous 2π periodic functions $\boldsymbol{x}(t)$, $\boldsymbol{x}(t) : \mathbb{R} \mapsto \mathbb{R}^N$.

Define the circle group S^1 such that any element θ acts as a change of phase, i.e.,

$$\theta \cdot \boldsymbol{x}(t) = \boldsymbol{x}(t - \theta).$$

Temporal symmetry

Let $C_{2\pi}$ be the space of continuous 2π periodic functions $\boldsymbol{x}(t)$, $\boldsymbol{x}(t) : \mathbb{R} \mapsto \mathbb{R}^N$.

Define the circle group S^1 such that any element θ acts as a change of phase, i.e.,

$$\theta \cdot \boldsymbol{x}(t) = \boldsymbol{x}(t - \theta).$$

This is an exact symmetry of elements of $C_{2\pi}$ and we construct a numerical method to preserve this symmetry.

Spectral solution

Construct a periodic solution of the form

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \sum_{m=1}^{M} \left(\boldsymbol{x}_m \exp(im\omega t) + \overline{\boldsymbol{x}_m} \exp(-im\omega t) \right)$$

where

$$\boldsymbol{x}_m = \boldsymbol{x}_m^{Re} + i\boldsymbol{x}_m^{Im}, \quad m = 1, \ldots, M \quad \text{and} \quad \omega = \frac{2\pi}{T} .$$

The number of unknowns in this description is N(2M+1)+1.

Differentiating w.r.t. time

$$\frac{d\boldsymbol{x}}{dt} = i\omega \sum_{m=1}^{M} m \left(\boldsymbol{x}_m \exp(im\omega t) - \overline{\boldsymbol{x}_m} \exp(-im\omega t) \right)$$

Collocation

Define

$$\hat{t}_i = \frac{(i-1)\omega}{2M+1} = \frac{2\pi(i-1)}{(2M+1)T}, \; i = 1, \ldots, 2M+1$$

and solve

$$\dot{\boldsymbol{x}}(\hat{t}_i) - \boldsymbol{f}(\boldsymbol{x}(\hat{t}_i), \lambda) = \mathbf{0}, \; i = 1, \ldots, 2M+1$$

for $\boldsymbol{x}_i, \; i = 0, \ldots, M$.

Augment with the phase condition.

Galerkin

Construct test functions

$$\boldsymbol{\psi}(t) = \boldsymbol{\psi}_0 + \sum_{n=1}^{M} \left(\boldsymbol{\psi}_n \exp(in\omega t) + \overline{\boldsymbol{\psi}_n} \exp(-in\omega t) \right),$$

where

$$\boldsymbol{\psi}_n = \boldsymbol{\psi}_n^{Re} + i\, \boldsymbol{\psi}_n^{Im} \quad \text{for} \quad n = 1, \ldots, M$$

and require

$$\int_0^T \left[(\dot{\boldsymbol{x}} - \boldsymbol{f}(\boldsymbol{x}(t), \lambda) \right] \boldsymbol{\psi}(t)\, dt = 0 \quad \forall\, \boldsymbol{\psi}(t). \tag{3}$$

Augment with the phase condition.

Galerkin

Notice that equations

$$\int_0^T \left[\dot{\boldsymbol{x}} - \boldsymbol{f}(\boldsymbol{x}(t), \lambda)\right] \boldsymbol{\psi}(t)\, dt = 0 \quad \forall\, \boldsymbol{\psi}(t)$$

simplify considerably owing to the orthogonality

$$\int_0^T \exp(in\omega t) \exp(-im\omega t)\, dt = 0 \quad \text{if} \quad m \neq n.$$

Qu: Where is the Monodromy matrix in all this?

Computation of periodic orbits of the N.-S. equations

Goal: To construct a numerical method to follow paths of periodic solutions in parameter space efficiently. It should converge rapidly near Hopf bifurcation points and near Takens-Bogdanov points where the period increases without bound (making simulation unattractive). It should be able to locate paths of unstable periodic orbits arising, for example, at subcritical Hopf bifurcation points.

The two-dimensional Navier-Stokes equations are

$$\left.\begin{aligned} R(\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) + \nabla p - \nabla^2\boldsymbol{u} &= 0 \\ \nabla\cdot\boldsymbol{u} &= 0 \end{aligned}\right\} \text{ in } \Omega, \quad \text{with} \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on} \quad \partial\Omega.$$

Computation of periodic orbits of the N.-S. equations

Our method is based upon standard finite-element of space coupled to a spectral discretization of time. We define

$$u(x,y,t) = u_0(x,y) + \sum_{m=1}^{M} \Big(u_m(x,y)\exp(im\omega t) + \overline{u_m}(x,y)\exp(-im\omega t) \Big)$$

where $u_m(x,y) = u_m^{Re}(x,y) + i u_m^{Im}(x,y), \quad m = 1,\ldots,M.$

Computation of periodic orbits of the N.-S. equations

Our method is based upon standard finite-element of space coupled to a spectral discretization of time. We define

$$u(x,y,t) = u_0(x,y) + \sum_{m=1}^{M} \left(u_m(x,y)\exp(im\omega t) + \overline{u_m}(x,y)\exp(-im\omega t) \right)$$

where $u_m(x,y) = u_m^{Re}(x,y) + iu_m^{Im}(x,y), \quad m = 1,\ldots,M.$

Similar expansions are used for $v(x,y,t)$ and $p(x,y,t)$.

Computation of periodic orbits of the N.-S. equations

Our method is based upon standard finite-element of space coupled to a spectral discretization of time. We define

$$u(x,y,t) = u_0(x,y) + \sum_{m=1}^{M} \Big(u_m(x,y)\exp(im\omega t) + \overline{u_m}(x,y)\exp(-im\omega t) \Big)$$

where $u_m(x,y) = u_m^{Re}(x,y) + iu_m^{Im}(x,y), \quad m = 1,\ldots,M.$

Similar expansions are used for $v(x,y,t)$ and $p(x,y,t)$.

The test functions have the form

$$\psi(x,y,t) = \psi_0(x,y) + \sum_{n=1}^{M} \Big(\psi_n(x,y)\exp(in\omega t) + \overline{\psi_n}(x,y)\exp(-in\omega t) \Big),$$

where $\psi_n(x,y) = \psi_n^{Re}(x,y) + i\,\psi_n^{Im}(x,y)$, for $n = 1,\ldots,M$, etc.

Computation of periodic orbits of the N.-S. equations

The Navier-Stokes equations in weak form (after differentiating by parts) are then

$$
\begin{aligned}
&\int_0^{2\pi/\omega}\int_\Omega \\
&\quad R(u_t + u u_x + v u_y)\,\psi - p\,\psi_x + (u_x\,\psi_x + u_y\,\psi_y) \\
&+R(v_t + u v_x + v v_y)\,\phi - p\,\phi_y + (v_x\,\phi_x + v_y\,\phi_y) \\
&+(u_x + v_y)\,\xi\; d\boldsymbol{x} dt = 0.
\end{aligned}
\tag{4}
$$

where the integral is taken over the domain Ω and a single period $[0, T] = [0, 2\pi/\omega]$.

Computation of periodic orbits of the N.-S. equations

Significant simplification of the integrals that arise when these expansions are substituted into the weak form (4), arises due to the observation that

$$\int_0^{2\pi/\omega} \exp(im\omega t)\exp(in\omega t)\, dt = 0 \text{ if } m \neq -n.$$

Computation of periodic orbits of the N.-S. equations

Significant simplification of the integrals that arise when these expansions are substituted into the weak form (4), arises due to the observation that

$$\int_0^{2\pi/\omega} \exp(im\omega t)\exp(in\omega t)\, dt = 0 \text{ if } m \neq -n.$$

We discretize velocity components $u_0,\ u_m, m = 1, \ldots, M$, etc. pressure and test function components using the same finite element mesh.

Computation of periodic orbits of the N.-S. equations

Significant simplification of the integrals that arise when these expansions are substituted into the weak form (4), arises due to the observation that

$$\int_0^{2\pi/\omega} \exp(im\omega t)\exp(in\omega t)\, dt = 0 \text{ if } m \neq -n.$$

We discretize velocity components u_0, $u_m, m = 1, \ldots, M$, etc. pressure and test function components using the same finite element mesh.

We set a phase condition by requiring the real part of the cross-stream velocity of the first Fourier mode to be zero at one point along the centerline.

Computation of periodic orbits of the N.-S. equations

For example, when $m = 1$, the contributions to the coefficients multiplying ψ_0, ψ_1^{Re} and ψ_1^{Im} from the nonlinear term uu_x alone are

$$\begin{aligned}
(\psi_0):& \qquad u_0 u_{0,x} + 2u_1^{Re} u_{1,x}^{Re} + 2u_1^{Im} u_{1,x}^{Im} \\
(\psi_1^{Re}):& \qquad 2(u_0 u_{1,x}^{Re} + u_1^{Re} u_{0,x}) \\
(\psi_1^{Im}):& \qquad -2(u_0 u_{1,x}^{Im} + u_1^{Im} u_{0,x})
\end{aligned}$$

respectively.

Computation of periodic orbits of the N.-S. equations

When $m = 2$, the contributions to the coefficients multiplying $\psi_0, \psi_1^{Re}, \psi_1^{Im}, \psi_2^{Re}$ and ψ_2^{Im} from the nonlinear term uu_x (alone) are

$$
\begin{aligned}
(\psi_0) : \quad & u_0 u_{0,x} + 2u_1^{Re} u_{1,x}^{Re} + 2u_1^{Im} u_{1,x}^{Im} \\
& +2u_2^{Re} u_{2,x}^{Re} + 2u_2^{Im} u_{2,x}^{Im} \\
(\psi_1^{Re}) : \quad & 2(u_0 u_{1,x}^{Re} + u_1^{Re} u_{0,x} \\
& +u_1^{Re} u_{2,x}^{Re} + u_2^{Re} u_{1,x}^{Re} + u_1^{Im} u_{2,x}^{Im} + u_2^{Im} u_{1,x}^{Re}) \\
(\psi_1^{Im}) : & -2(u_0 u_{1,x}^{Im} + u_1^{Im} u_{0,x} \\
& +u_1^{Re} u_{2,x}^{Im} + u_2^{Im} u_{1,x}^{Re} - u_1^{Im} u_{2,x}^{Re} - u_2^{Im} u_{1,x}^{Re}) \\
(\psi_2^{Re}) : \quad & 2(u_0 u_{2,x}^{Re} + u_2^{Re} u_{0,x} + u_1^{Re} u_{1,x}^{Re} - u_1^{Im} u_{1,x}^{Im}) \\
(\psi_2^{Im}) : & -2(u_0 u_{2,x}^{Im} + u_2^{Im} u_{0,x} + u_1^{Im} u_{1,x}^{Re} + u_1^{Im} u_{1,x}^{Re})
\end{aligned}
$$

respectively.

Energy per temporal mode, B=0.7, Re=141

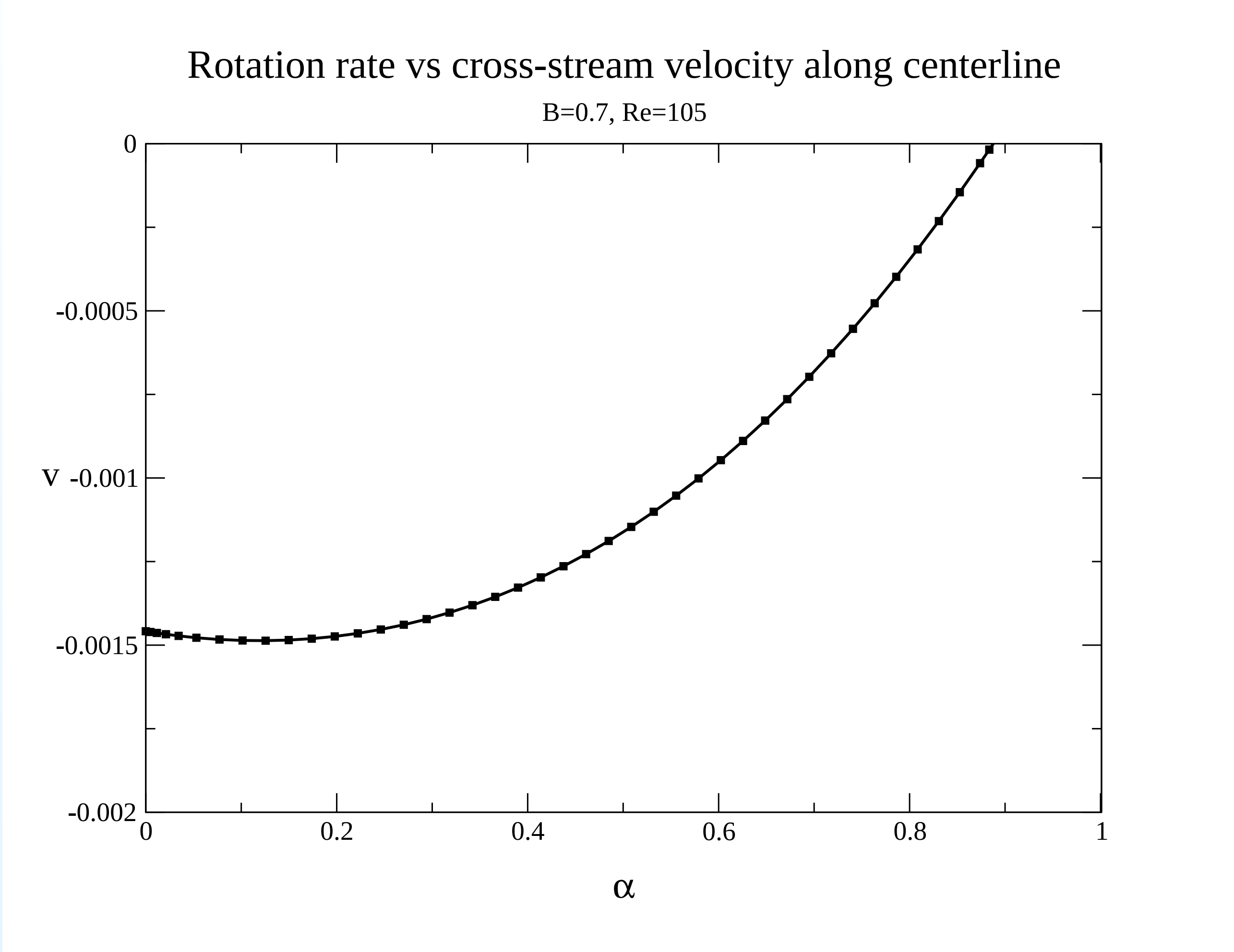
Define

$$
\begin{aligned}
E &= \frac{\omega}{4\pi}\int_0^{2\pi/\omega}\int_\Omega \boldsymbol{u}\cdot\boldsymbol{u}\,d\boldsymbol{x}dt \\
&= \frac{1}{2}\int_\Omega \boldsymbol{u}_0\cdot\boldsymbol{u}_0\,d\boldsymbol{x} + \sum_{m=1}^{M}\int_\Omega \boldsymbol{u}_m\cdot\overline{\boldsymbol{u}}_m\,d\boldsymbol{x}
\end{aligned}
$$

Energy per temporal mode, B=0.7, Re=141

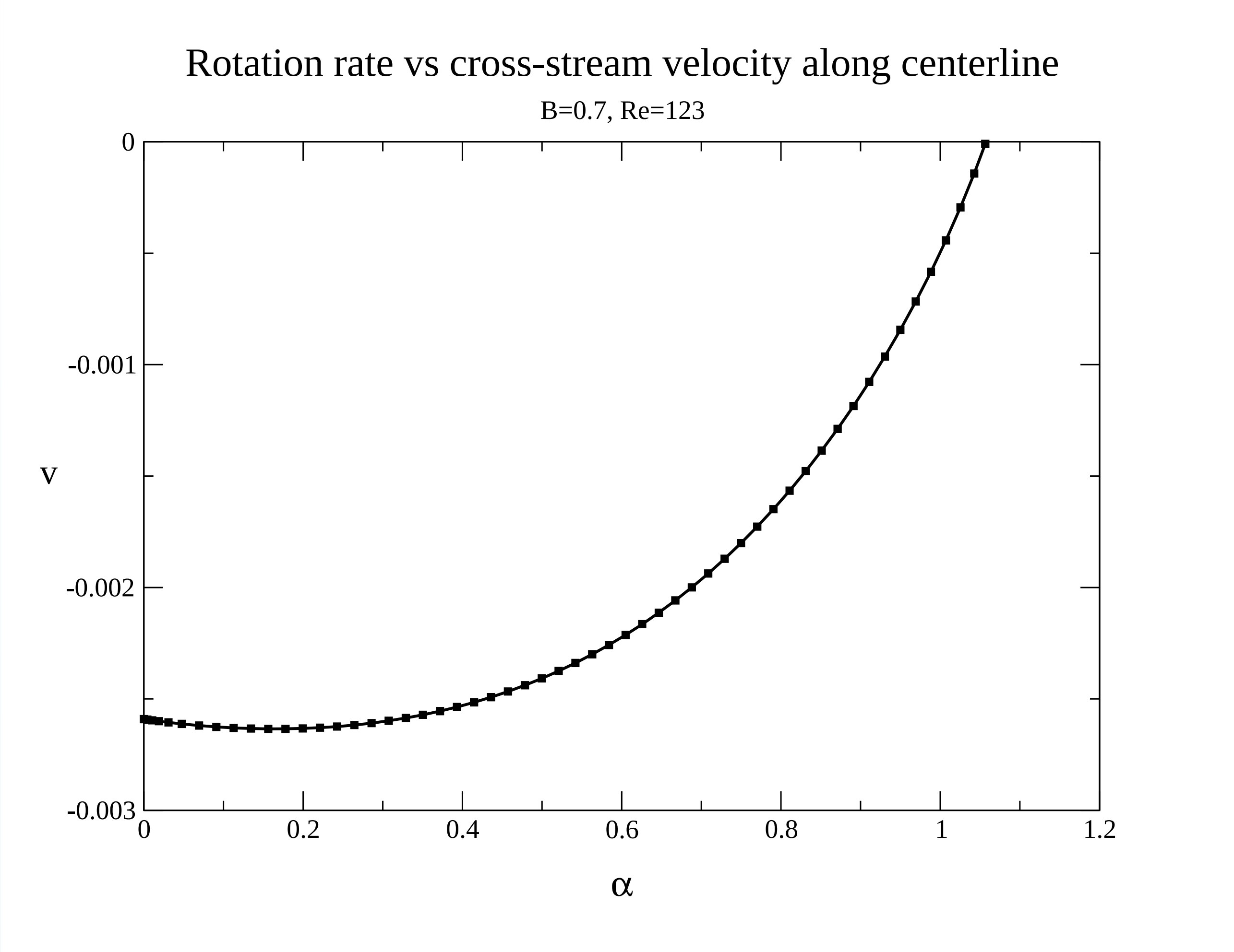
# modes	mode	energy
1	0	1.9557E+03
	1	6.7497E+01
4	0	1.9562E+03
	1	6.4696E+01
	2	3.5207E+00
	3	3.4594E-01
	4	2.1614E-02
7	0	1.9562E+03
	1	6.4711E+01
	2	3.5357E+00
	3	3.3695E-01
	4	1.7885E-02
	5	1.6529E-03
	6	1.5099E-04
	7	2.2889E-05

B=0.7: v(centerline) vs α



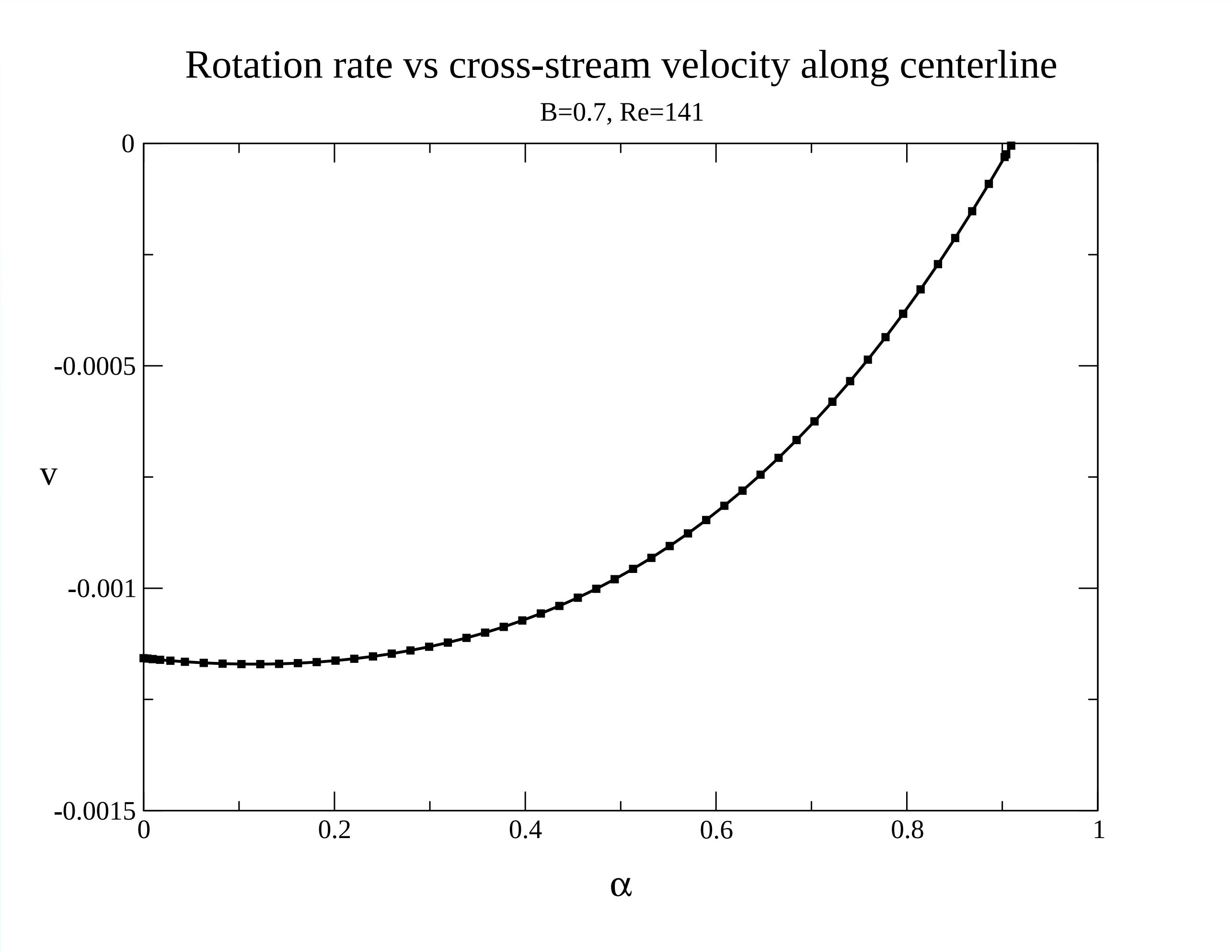
Cross-stream velocity of mode 2 at the centerline of the channel as a function of α for $R = 105$

B=0.7: v(centerline) vs α



Cross-stream velocity of mode 2 at the centerline of the channel as a function of α for $R = 123$

B=0.7: v(centerline) vs α



Cross-stream velocity of mode 2 at the centerline of the channel as a function of α for $R = 141$.

Reflectional symmetry $(\alpha = 0)$

Consider the nonlinear equation

$$\boldsymbol{f}(\boldsymbol{x};\lambda) = \mathbf{0}; \quad \boldsymbol{f} : \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{R}^N.$$

Reflectional symmetry $(\alpha = 0)$

Consider the nonlinear equation

$$\boldsymbol{f}(\boldsymbol{x};\lambda) = \boldsymbol{0}; \quad \boldsymbol{f} : \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{R}^N.$$

A non-singular $(N \times N)$ matrix $\boldsymbol{S}$ is a reflectional (Z_2) symmetry of these equations if

$$\boldsymbol{f}(\boldsymbol{S}\boldsymbol{x};\lambda) = \boldsymbol{S}\boldsymbol{f}(\boldsymbol{x};\lambda), \quad \text{where} \quad \boldsymbol{S} \neq \boldsymbol{I}, \quad \boldsymbol{S}^2 = \boldsymbol{I}.$$

Reflectional symmetry $(\alpha = 0)$

Consider the nonlinear equation

$$\boldsymbol{f}(\boldsymbol{x};\lambda) = \boldsymbol{0}; \quad \boldsymbol{f} : \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{R}^N.$$

A non-singular $(N \times N)$ matrix $\boldsymbol{S}$ is a reflectional (Z_2) symmetry of these equations if

$$\boldsymbol{f}(\boldsymbol{S}\boldsymbol{x};\lambda) = \boldsymbol{S}\boldsymbol{f}(\boldsymbol{x};\lambda), \quad \text{where} \quad \boldsymbol{S} \neq \boldsymbol{I}, \quad \boldsymbol{S}^2 = \boldsymbol{I}.$$

At a Hopf bifurcation point where $(\boldsymbol{f}_{\boldsymbol{x}}\boldsymbol{x} + i\omega\boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}$, the equivariance (symmetry) of $\boldsymbol{f}$ with respect to $\boldsymbol{S}$ implies

$$\boldsymbol{S}\boldsymbol{v} = \pm\boldsymbol{v}.$$

We call $\boldsymbol{S}\boldsymbol{v} = -\boldsymbol{v}$ the (Z_2) symmetry-breaking case.

Reflectional symmetry $(\alpha = 0)$

Let $\phi(t)$ be the unique periodic orbit guaranteed by the Hopf bifurcation theorem to bifurcate from Hopf bifurcation point, with period T.

Reflectional symmetry $(\alpha = 0)$

Let $\phi(t)$ be the unique periodic orbit guaranteed by the Hopf bifurcation theorem to bifurcate from Hopf bifurcation point, with period T.

In the (Z_2) symmetry-breaking case

$$\begin{aligned} \boldsymbol{\phi}(t) &= \boldsymbol{\phi}(t+T) \\ \boldsymbol{S}\boldsymbol{\phi}(t) &\neq \boldsymbol{\phi}(t) \end{aligned}$$

Reflectional symmetry $(\alpha = 0)$

Let $\phi(t)$ be the unique periodic orbit guaranteed by the Hopf bifurcation theorem to bifurcate from Hopf bifurcation point, with period T.

In the (Z_2) symmetry-breaking case

$$\begin{aligned} \boldsymbol{\phi}(t) &= \boldsymbol{\phi}(t+T) \\ \boldsymbol{S}\boldsymbol{\phi}(t) &\neq \boldsymbol{\phi}(t) \end{aligned}$$

Rather

$$\boldsymbol{S}\boldsymbol{\phi}(t) = \boldsymbol{\phi}\left(t + \frac{T}{2}\right)$$

Reflectional symmetry $(\alpha = 0)$

Let $\phi(t)$ be the unique periodic orbit guaranteed by the Hopf bifurcation theorem to bifurcate from Hopf bifurcation point, with period T.

In the (Z_2) symmetry-breaking case

$$\begin{aligned} \boldsymbol{\phi}(t) &= \boldsymbol{\phi}(t+T) \\ \boldsymbol{S}\boldsymbol{\phi}(t) &\neq \boldsymbol{\phi}(t) \end{aligned}$$

Rather

$$\boldsymbol{S}\boldsymbol{\phi}(t) = \boldsymbol{\phi}\left(t + \frac{T}{2}\right)$$

Savings of 2-8 can be obtained by performing computations on one half of the computational domain.

Proof

$$
\begin{aligned}
\dot{\boldsymbol{\phi}}(t) + \boldsymbol{f}(\boldsymbol{\phi}(t), \lambda) &= \mathbf{0}, \\
\boldsymbol{S}\dot{\boldsymbol{\phi}}(t) + \boldsymbol{S}\boldsymbol{f}(\boldsymbol{\phi}(t), \lambda) &= \mathbf{0}, \\
\boldsymbol{S}\dot{\boldsymbol{\phi}}(t) + \boldsymbol{f}(\boldsymbol{S}\boldsymbol{\phi}(t), \lambda) &= \mathbf{0},
\end{aligned}
$$

i.e., $\boldsymbol{S}\boldsymbol{\phi}(t)$ is also a nontrivial periodic solution.

Proof

$$\begin{aligned} \dot{\boldsymbol{\phi}}(t) + \boldsymbol{f}(\boldsymbol{\phi}(t), \lambda) &= \mathbf{0}, \\ \boldsymbol{S}\dot{\boldsymbol{\phi}}(t) + \boldsymbol{S}\boldsymbol{f}(\boldsymbol{\phi}(t), \lambda) &= \mathbf{0}, \\ \boldsymbol{S}\dot{\boldsymbol{\phi}}(t) + \boldsymbol{f}(\boldsymbol{S}\boldsymbol{\phi}(t), \lambda) &= \mathbf{0}, \end{aligned}$$

i.e., $\boldsymbol{S}\boldsymbol{\phi}(t)$ is also a nontrivial periodic solution.

Let

$$\boldsymbol{S}\boldsymbol{\phi}(t) = \boldsymbol{\phi}(t+\rho), \quad \rho \neq 0.$$

But

$$\boldsymbol{\phi}(t) = \boldsymbol{S}^2\boldsymbol{\phi} = \boldsymbol{\phi}(t+2\rho)$$

hence $2\rho = nT$ or $\rho = nT/2$, where n must be odd, otherwise

$$\boldsymbol{S}\boldsymbol{\phi}(t) = \boldsymbol{\phi}(t+\rho) = \boldsymbol{\phi}(t+nT/2) = \boldsymbol{\phi}(t).$$

Proof

$$
\begin{aligned}
\dot{\boldsymbol{\phi}}(t) + \boldsymbol{f}(\boldsymbol{\phi}(t), \lambda) &= \mathbf{0}, \\
\boldsymbol{S}\dot{\boldsymbol{\phi}}(t) + \boldsymbol{S}\boldsymbol{f}(\boldsymbol{\phi}(t), \lambda) &= \mathbf{0}, \\
\boldsymbol{S}\dot{\boldsymbol{\phi}}(t) + \boldsymbol{f}(\boldsymbol{S}\boldsymbol{\phi}(t), \lambda) &= \mathbf{0},
\end{aligned}
$$

i.e., $\boldsymbol{S}\boldsymbol{\phi}(t)$ is also a nontrivial periodic solution.

Let

$$\boldsymbol{S}\boldsymbol{\phi}(t) = \boldsymbol{\phi}(t+\rho), \quad \rho \neq 0.$$

But

$$\boldsymbol{\phi}(t) = \boldsymbol{S}^2\boldsymbol{\phi} = \boldsymbol{\phi}(t+2\rho)$$

hence $2\rho = nT$ or $\rho = nT/2$, where n must be odd, otherwise

$$\boldsymbol{S}\boldsymbol{\phi}(t) = \boldsymbol{\phi}(t+\rho) = \boldsymbol{\phi}(t+nT/2) = \boldsymbol{\phi}(t).$$

Implications of reflectional symmetry

Construct a periodic solution of the form

$$\begin{aligned}
\boldsymbol{S}\boldsymbol{x}(t) &= \boldsymbol{S}\left(\boldsymbol{x}_0 + \sum_{m=1}^{M}\left(\boldsymbol{x}_m \exp(im\omega t) + \overline{\boldsymbol{x}_m}\exp(-im\omega t)\right)\right) \\
&= \boldsymbol{x}_0 + \sum_{m=1}^{M}\left(\boldsymbol{x}_m \exp(im\omega(t+\pi/\omega)) + \overline{\boldsymbol{x}_m}\exp(-im\omega(t+\pi/\omega))\right) \\
&= \boldsymbol{x}_0 + \sum_{m=1}^{M}\left((-1)^m\boldsymbol{x}_m \exp(im\omega t) + (-1)^m\overline{\boldsymbol{x}_m}\exp(-im\omega t)\right)
\end{aligned}$$

Implications of reflectional symmetry

Construct a periodic solution of the form

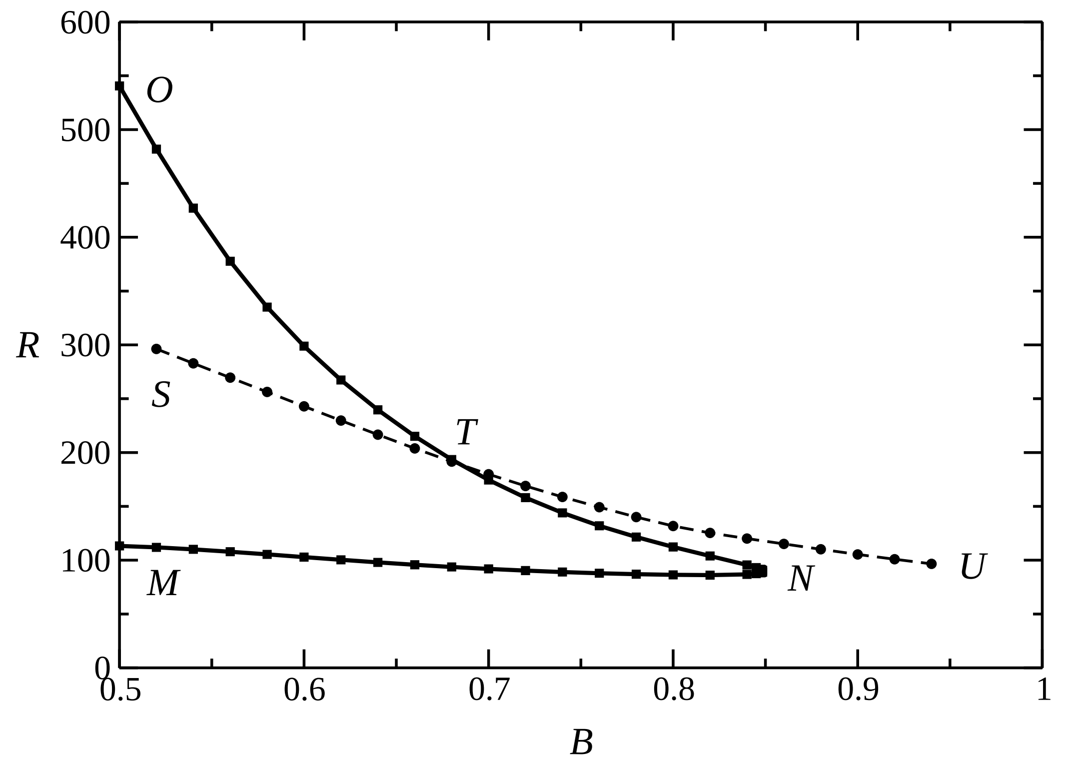
$$\begin{aligned}
\boldsymbol{S}\boldsymbol{x}(t) &= \boldsymbol{S}\left(\boldsymbol{x}_0 + \sum_{m=1}^{M}\left(\boldsymbol{x}_m \exp(im\omega t) + \overline{\boldsymbol{x}_m}\exp(-im\omega t)\right)\right) \\
&= \boldsymbol{x}_0 + \sum_{m=1}^{M}\left(\boldsymbol{x}_m \exp(im\omega(t+\pi/\omega)) + \overline{\boldsymbol{x}_m}\exp(-im\omega(t+\pi/\omega))\right) \\
&= \boldsymbol{x}_0 + \sum_{m=1}^{M}\left((-1)^m\boldsymbol{x}_m \exp(im\omega t) + (-1)^m\overline{\boldsymbol{x}_m}\exp(-im\omega t)\right)
\end{aligned}$$

Therefore

$$\begin{aligned}
\boldsymbol{S}\boldsymbol{x}_m(t) &= \boldsymbol{x}_m \quad \text{if } m \text{ is even} \\
\boldsymbol{S}\boldsymbol{x}_m(t) &= -\boldsymbol{x}_m \quad \text{if } m \text{ is odd}
\end{aligned}$$

and we need to discretize only one half of the domain.

Steady symmetry breaking



Loci of singular points as a function blockage ratio for a non-rotating cylinder, i.e., $\alpha = 0$. The solid line $MNTO$ is a path of Hopf bifurcation points. The dashed line STU is a path of symmetry-breaking bifurcation points on the steady, symmetric branch. Their intersection, T, is a codimension-two point.

The effect of symmetry breaking

Streamlines of the steady asymmetric flow at $B = 0.9, \alpha = 0, R = 114.3$. Contours of the streamfunction are equally spaced.

Sahin and Owens *Phys. Fluids* **16**, 2004.
Kumar and Mittal *Int. J. Numer. Meth. Fluids* **50**, 2006.

Future Directions

1. Floquet multipliers: Very special structure.
2. Estimating errors in eigenvalues and adaptivity: using ideas from *a posteriori* analysis for coupled physics problems.

Spectral Collocation for the Van der Pol system

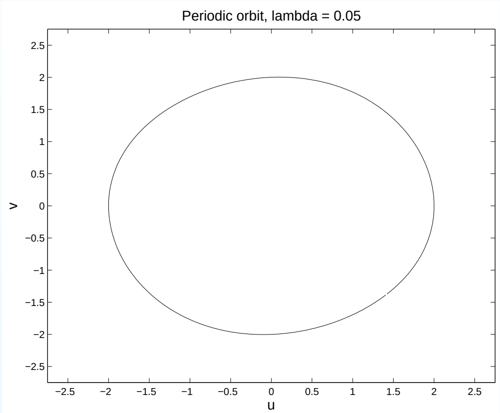
We wish to compute periodic solutions of the Van der Pol equation

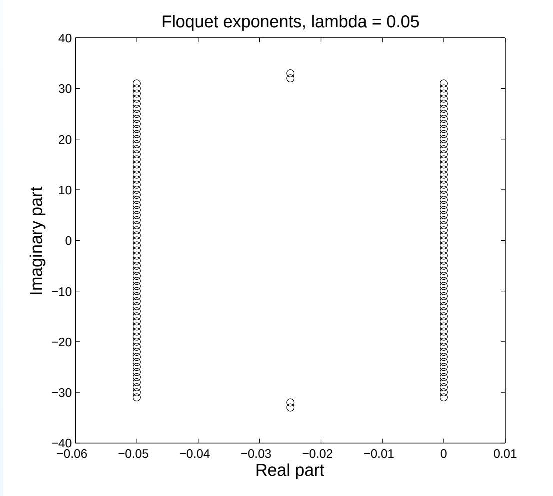
$$\ddot{u} - \lambda(1 - u^2)\dot{u} + u = 0. \tag{5}$$

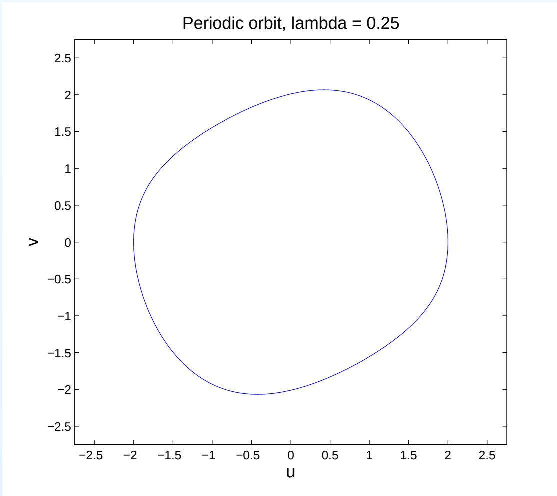
Writing the Van der Pol equation as a coupled system of first order differential equations, we have

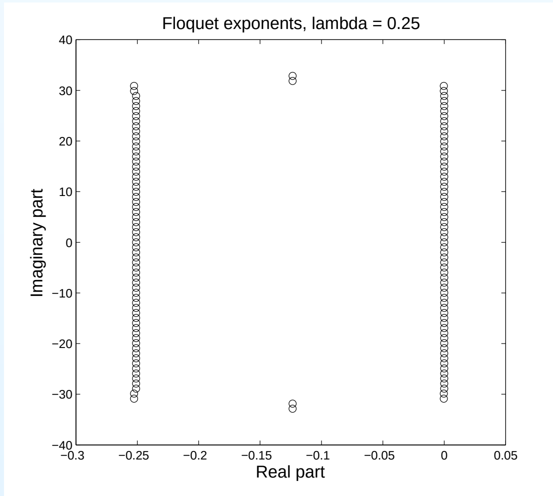
$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ \lambda(1 - u^2)v - u \end{pmatrix}. \tag{6}$$

Spectra for Van der Pol oscillator, $K = L = 32$

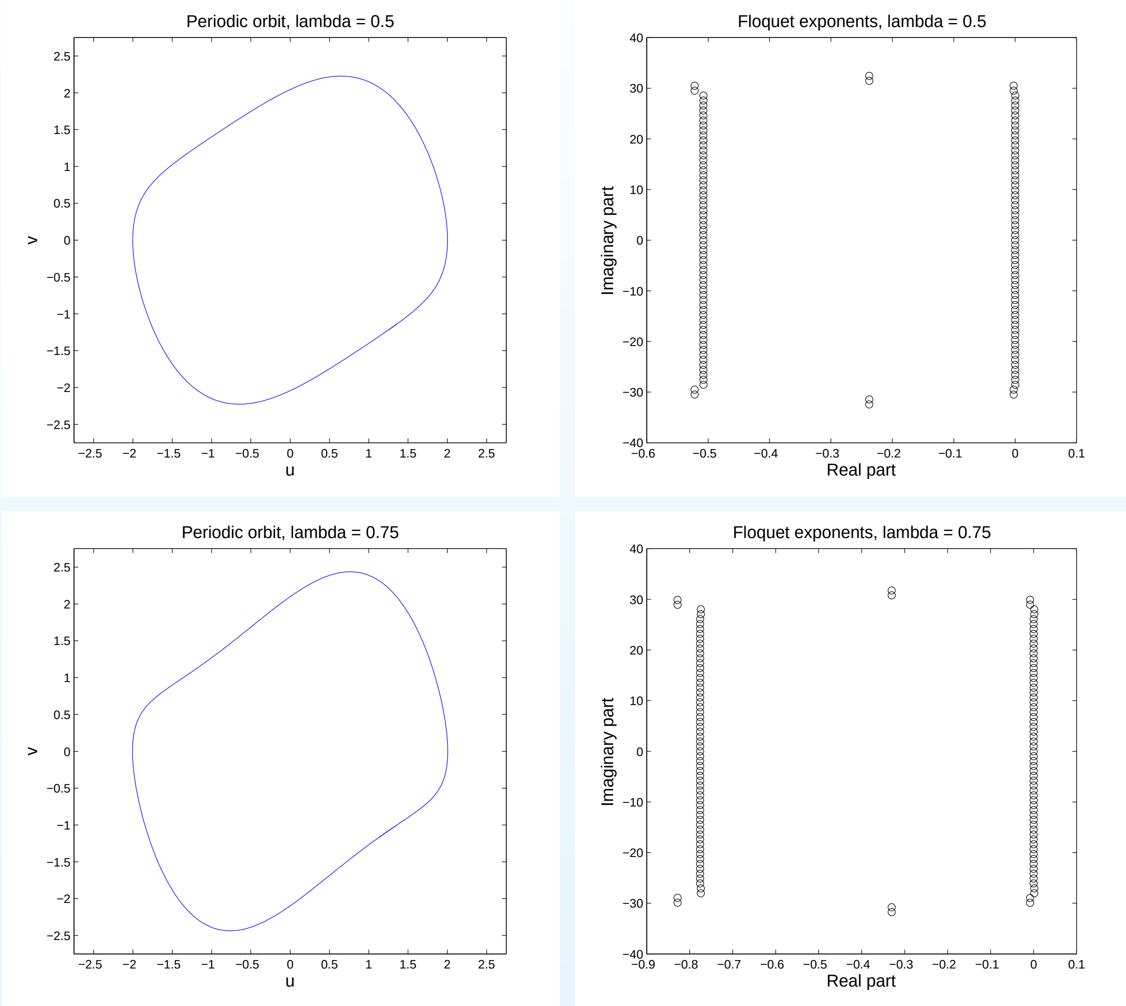








Spectra for Van der Pol oscillator, $K = L = 32$



Philosophy

- We wish to estimate the error in the solution of a multi-physics problem form

$$
\begin{aligned}
\mathcal{F}_1(x, u_1, Du_1, \ldots, u_n, Du_n) &= 0\,, \\
&\vdots \\
\mathcal{F}_n(x, u_1, Du_1, \ldots, u_n, Du_n) &= 0\,.
\end{aligned}
$$

Philosophy

- We wish to estimate the error in the solution of a multi-physics problem form

$$\begin{aligned}\mathcal{F}_1(x, u_1, Du_1, \ldots, u_n, Du_n) &= 0\,,\\ &\vdots\\ \mathcal{F}_n(x, u_1, Du_1, \ldots, u_n, Du_n) &= 0\,.\end{aligned}$$

- We can estimate the error if we can form the global adjoint of this system.

Philosophy

- We wish to estimate the error in the solution of a multi-physics problem form

$$\begin{aligned} \mathcal{F}_1(x, u_1, Du_1, \ldots, u_n, Du_n) &= 0\,, \\ &\vdots \\ \mathcal{F}_n(x, u_1, Du_1, \ldots, u_n, Du_n) &= 0\,. \end{aligned}$$

- We can estimate the error if we can form the global adjoint of this system.
- Assume that a discretization of the entire system is infeasible. Can we compute the error based on a combination of single physics residuals and *transfer errors*?

Philosophy

- We wish to estimate the error in the solution of a multi-physics problem form

$$\mathcal{F}_1(x, u_1, Du_1, \ldots, u_n, Du_n) = 0\,,$$
$$\vdots$$
$$\mathcal{F}_n(x, u_1, Du_1, \ldots, u_n, Du_n) = 0\,.$$

- We can estimate the error if we can form the global adjoint of this system.
- Assume that a discretization of the entire system is infeasible. Can we compute the error based on a combination of single physics residuals and *transfer errors*?
- Our strategy will be to represent transfer errors as quantities of interest and formulate the appropriate *single physics* adjoint problems.

Example Problem $\Omega_1 = \Omega_2 = \Omega$

$$
\begin{aligned}
-\Delta u_1 &= \sin(4\pi x)\sin(\pi y), \quad u_1 = 0 \;\text{ on } \partial\Omega \\
-\Delta u_2 &= \mathbf{b}\cdot\nabla u_1, \quad u_2 = 0 \text{ on } \partial\Omega \quad \text{where} \quad \mathbf{b} = \frac{2}{\pi}\begin{bmatrix} 25\sin(4\pi x) \\ \sin(\pi x) \end{bmatrix}.
\end{aligned}
$$

The corresponding adjoint problem is

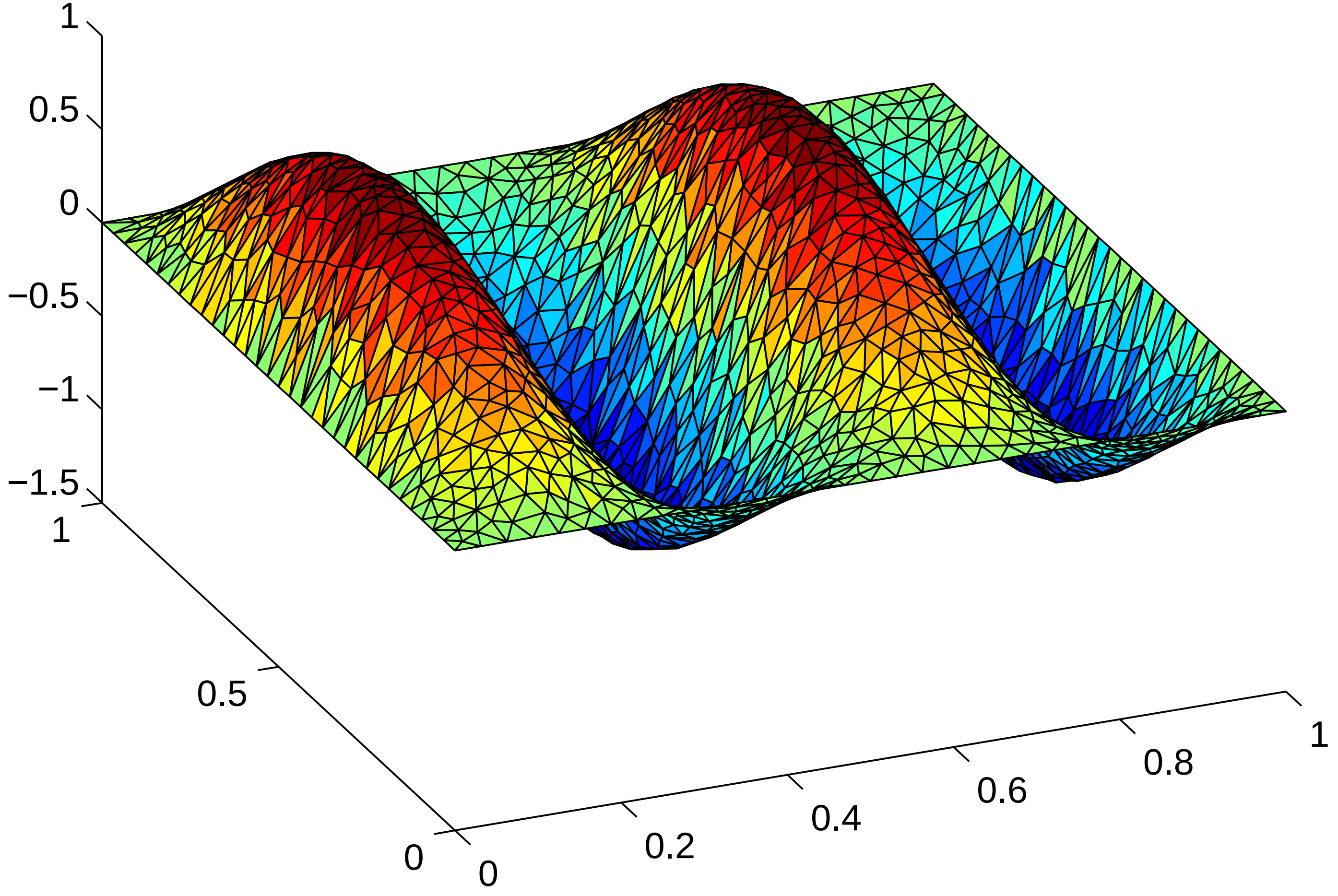
$$
\begin{aligned}
-\Delta\phi_1^{(1)} + Lf(u_1)\phi_2^{(1)} &= 0, \qquad \phi_1^{(1)} = 0 \;\text{ on } \partial\Omega \\
-\Delta\phi_2^{(1)} &= \delta_{\mathrm{reg}}^{x}, \quad \phi_2^{(1)} = 0 \text{ on } \partial\Omega\,.
\end{aligned}
$$

The secondary adjoint problem is

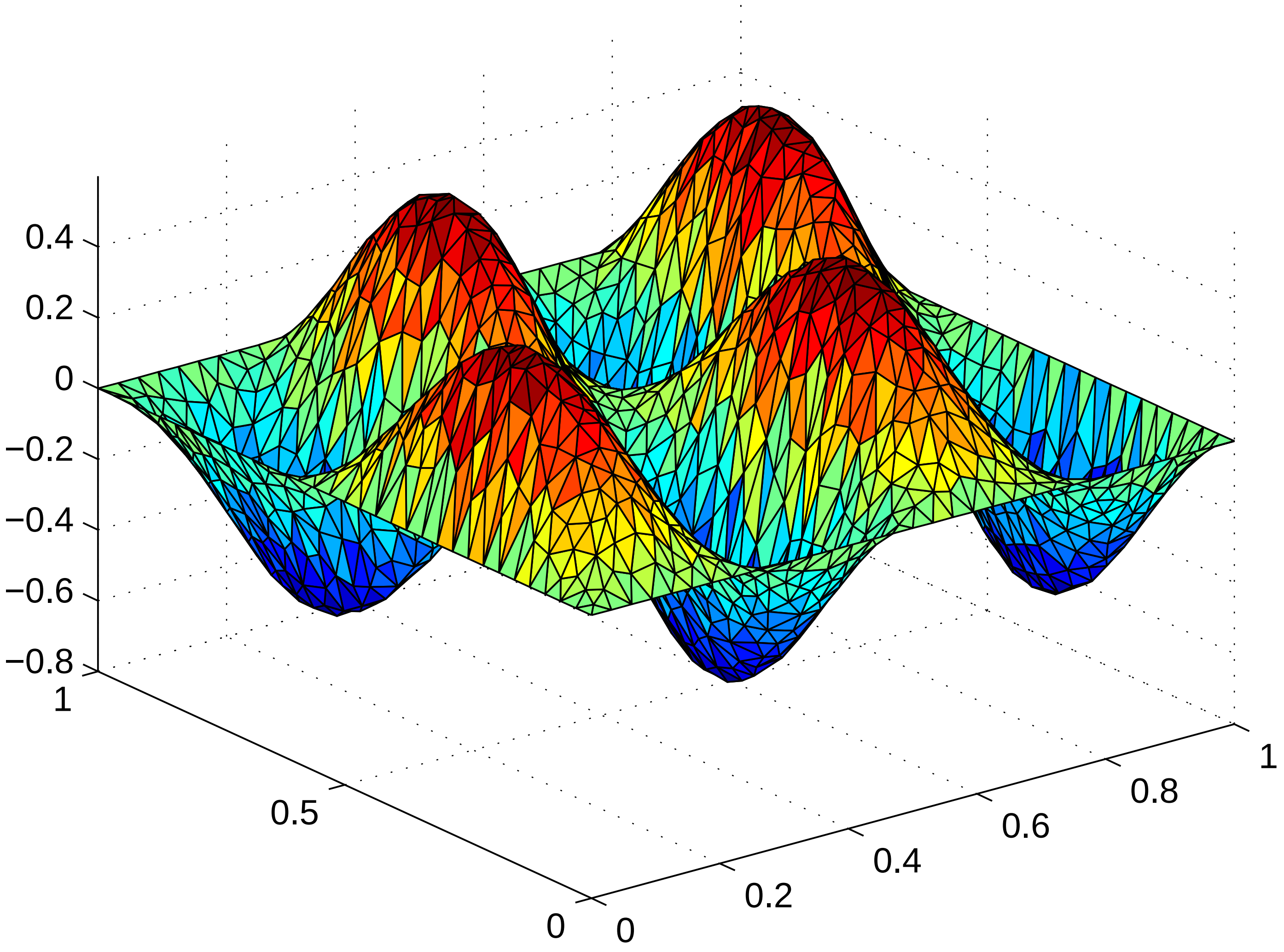
$$
\Delta\phi_1^{(2)} = \nabla\cdot(\mathbf{b}\phi_2^{(1)}), \quad \phi_1^{(2)} = 0 \text{ on } \partial\Omega\,.
$$

In the examples, linear elements were used to solve the forward problem, quadratic elements to solve the dual problem and x was chosen to be the point (.25,.25).

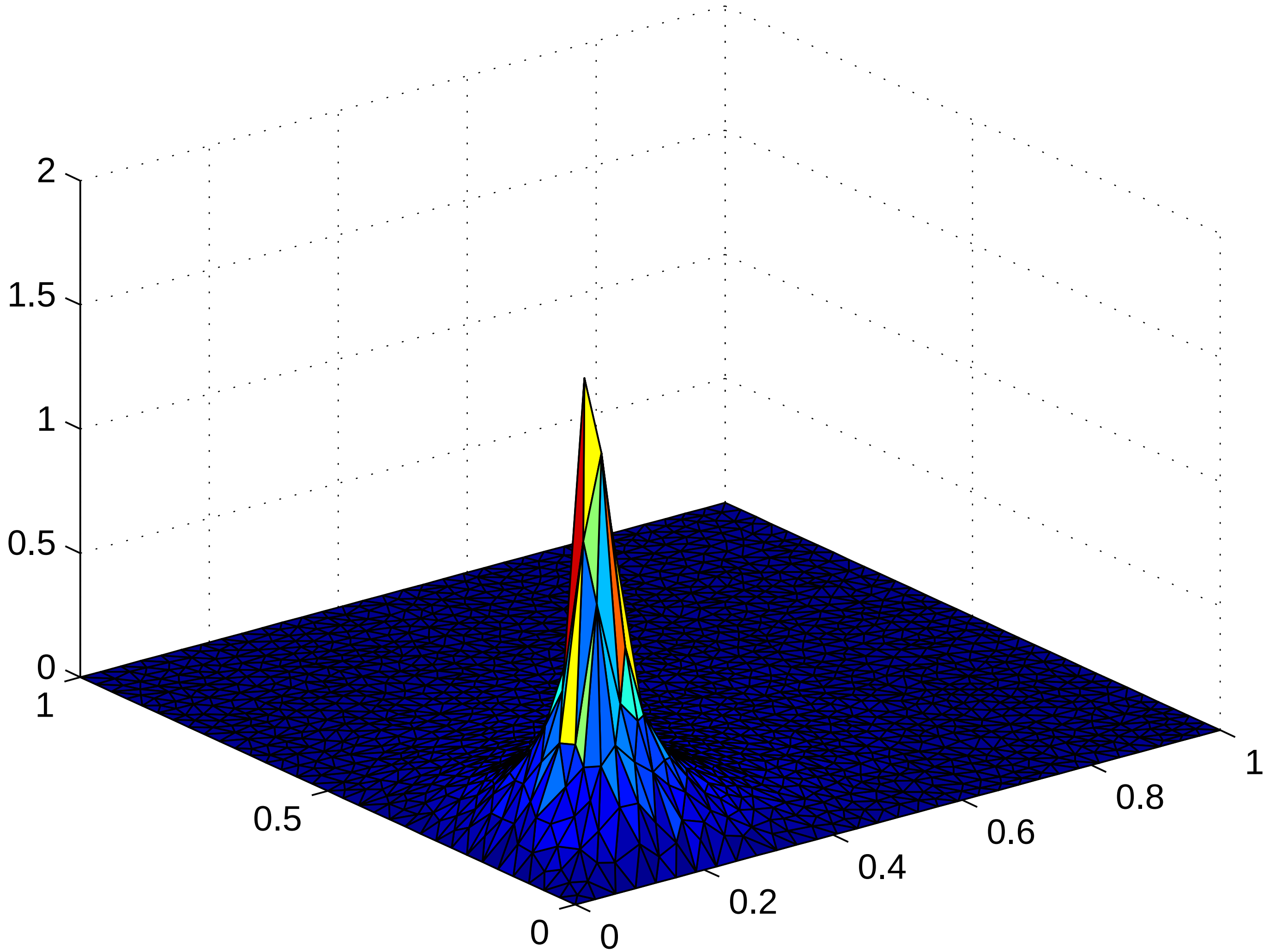
Model Problem: $u_{1,h}$ - fine mesh, no adaptivity



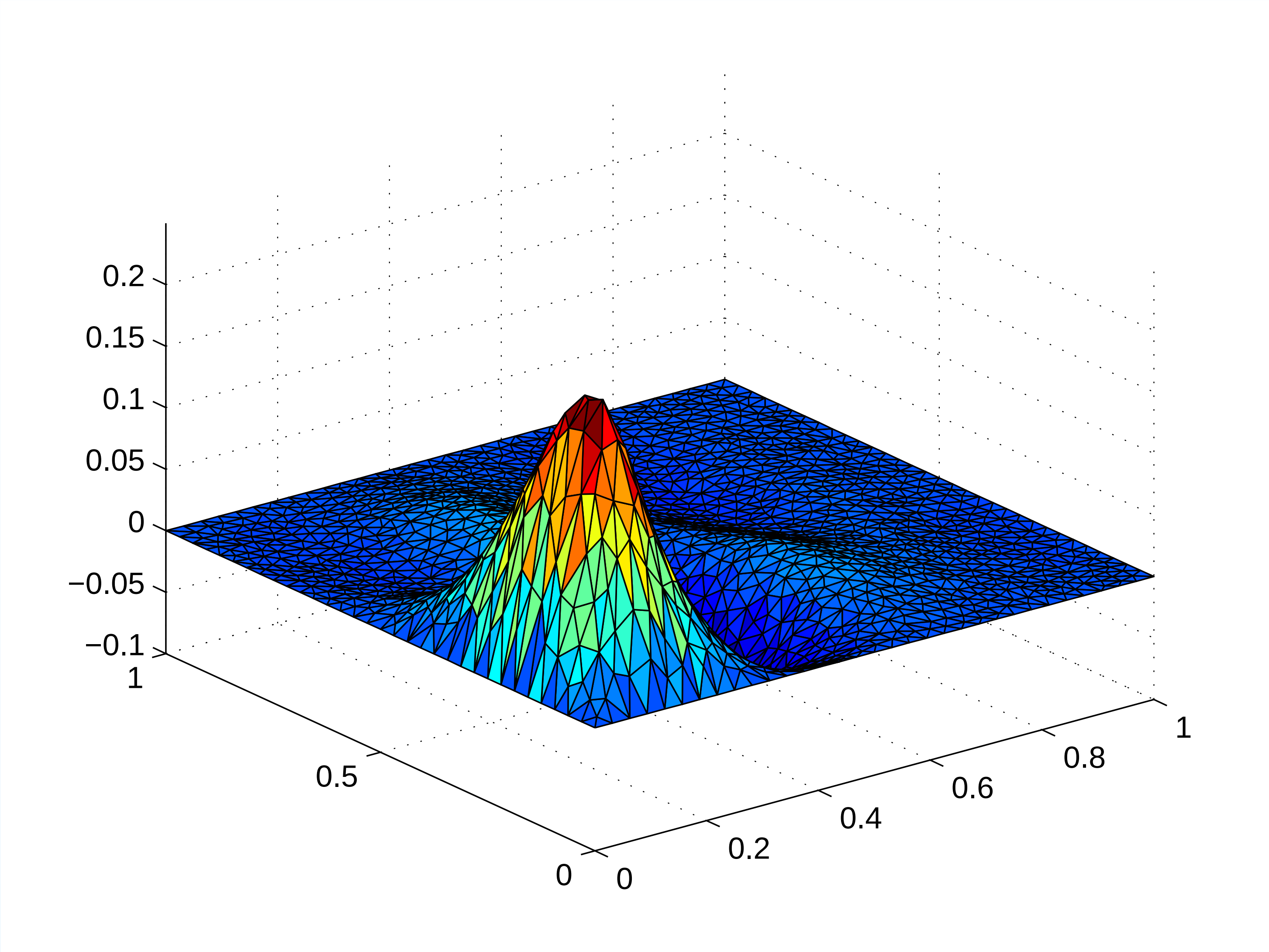
Model Problem: $u_{2,h}$ - fine mesh, no adaptivity



Model Problem: $\phi_{2,h}^{(1)}$ - fine mesh, no adaptivity

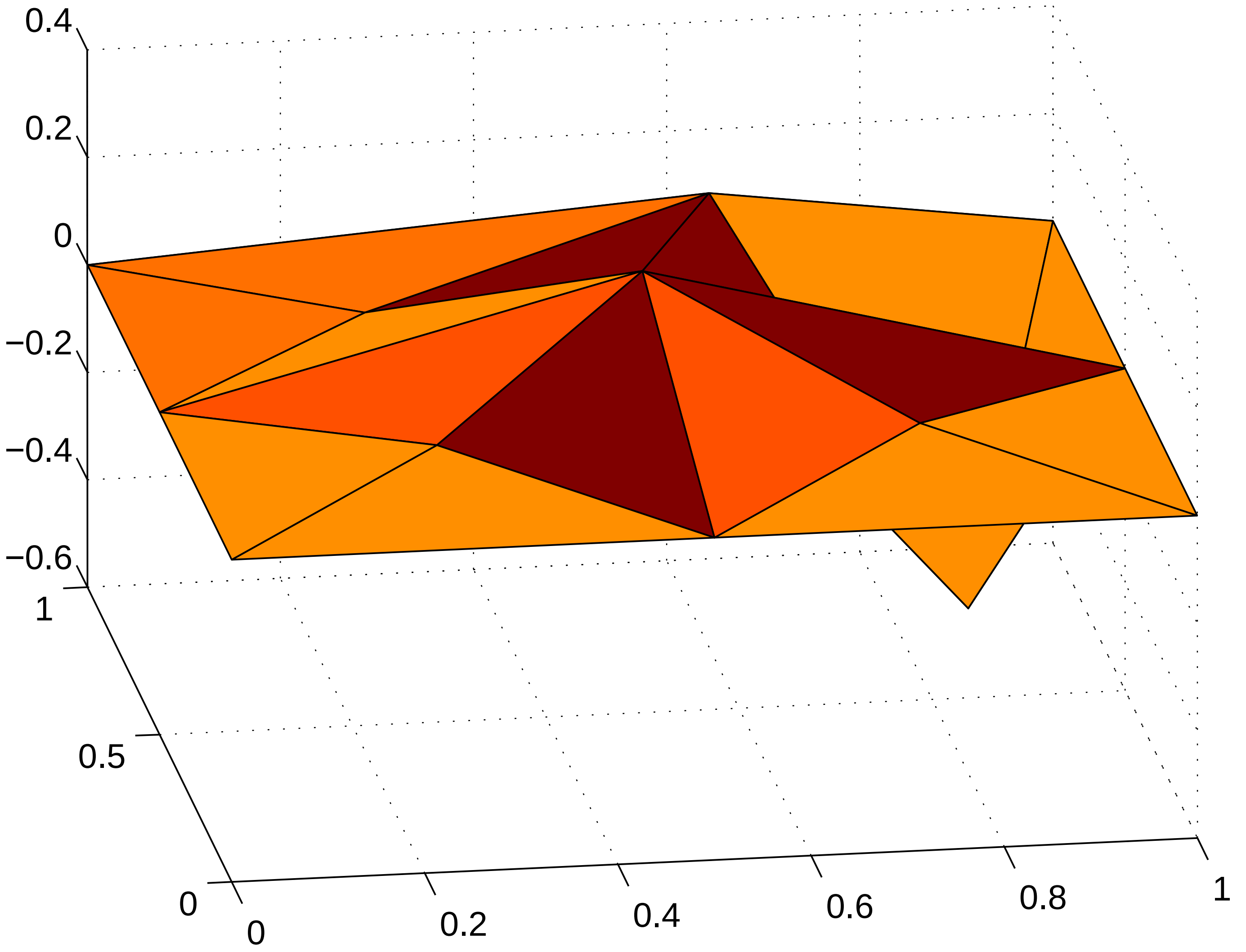


Model Problem: $\phi_{1,h}^{(2)}$ - fine mesh, no adaptivity

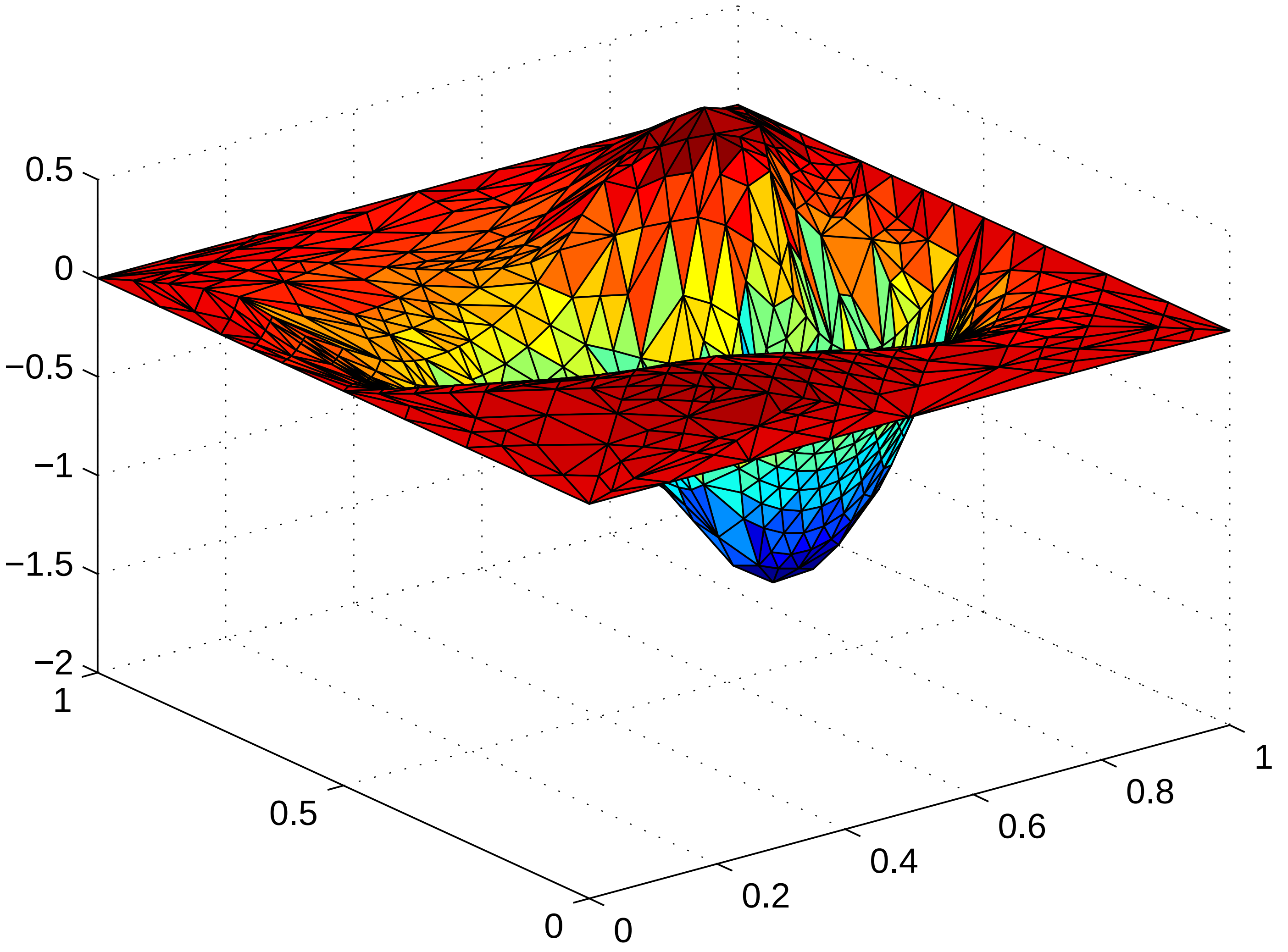


"Residual" error $(\mathcal{R}_2, \phi_2^{(1)}) \approx .0042$, "Transfer" error $(\mathcal{R}_1, \phi_1^{(2)}) \approx .0006$.

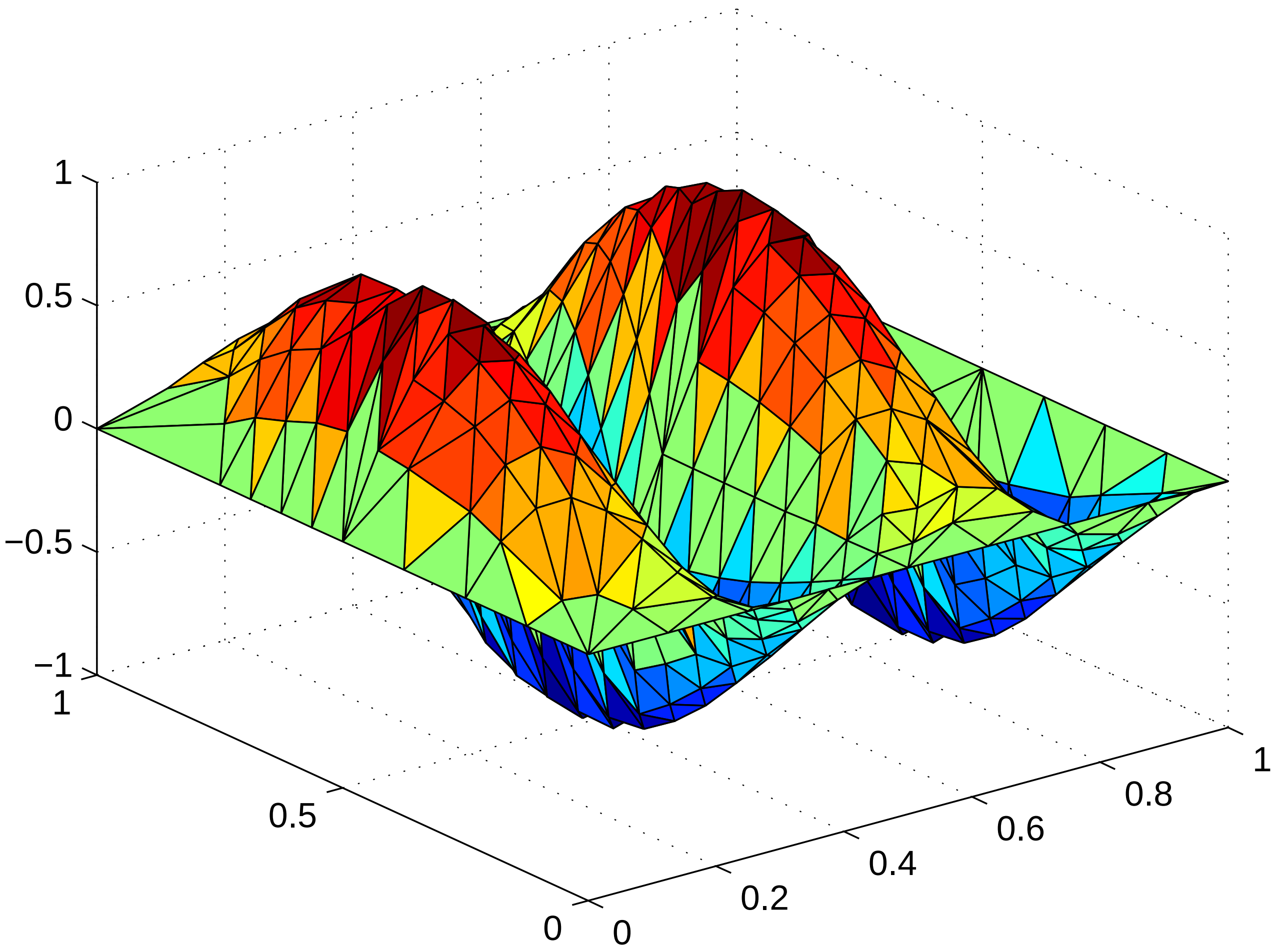
$u_{1,h}$ - coarse mesh



$u_{2,h}$ - $\psi = 1$ (“Garbage”)



$u_{1,h}$ - Adapt meshes independently ($\psi = 1$)



Note that the *gradient* of u_1 is transferred!

$u_{2,h}$ - Adapt meshes independently ($\psi = 1$)