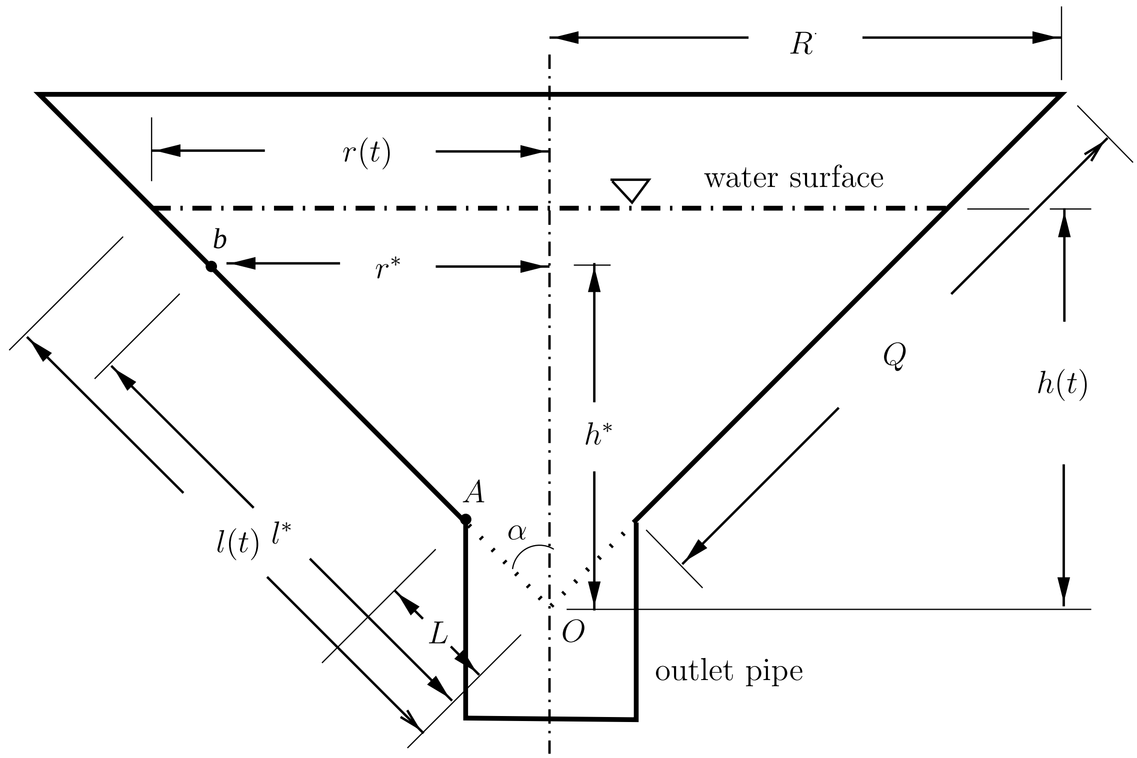
Related Rates

`Introduction:` Consider water draining from the bottom of a circular cylinder. The volume of water remaining in the cylinder is given by $v = \pi r^2 h$, where r is constant and h is the depth of water in the cylinder. Both v and h are functions of time and it is straightforward to show that $dv/dt = \pi r^2 dh/dt$. Now consider water draining from the bottom of a cone. The volume of water remaining in the cone is given by $v = \frac{1}{3}\pi r^2 h$, where r is a function of h which is a function of time. In this case the expression for dv/dt is not so obvious.

A. Apparatus

1. funnel
2. stand with ring support
3. water
4. ruler
5. masking tape
6. stopwatch
7. calibrated beaker
8. plastic overflow tray
9. tubing
10. tubing clamp
11. marker

B. Diagram



Experimental apparatus

C. Nomenclature

Capital letters denote fixed points or constant values determined by the geometry of the cone. Lower case letters denote values that change with time or that you select.

1. α = interior cone angle = 0.505 radians.
2. O = the projected point of intersection of the sides of the funnel.
3. A = the junction of the sides of the funnel and the outlet pipe.
4. L = the distance between O and A parallel to the sides of the funnel.
5. Q = the distance between A and the top of the funnel parallel to the sides of the funnel.
6. R = the radius of the top of the funnel.
7. t = time.
8. $v(t)$ = volume of the water in the funnel.
9. $l(t)$ = the distance between O and the water surface parallel to the sides of the funnel.
10. $h(t)$ = the vertical distance between O and the water surface.
11. $r(t)$ = the radius of the (circular) water surface.
12. b = chosen water surface location.
13. $l^\star$ = the distance between O and b parallel to the sides of the funnel.
14. $h^\star$ = the vertical distance between O and b.
15. $r^\star$ = the radius of the water surface at b.
16. Δv = a small change in volume.
17. Δt = an interval of time.
18. Δl = an interval of length parallel to the sides of the funnel.
19. dv/dt = rate of change of volume with time calculated using the chain rule.
20. ϵ = relative error.

D. Equations

1. $L = (R/\sin\alpha) - Q$.
2. Measured average rate of change of volume with time
$$\frac{\Delta v}{\Delta t} = \frac{\text{collected volume}}{\text{time elapsed during drainage}} \ .$$
3. Volume,
$$v(t) = \frac{1}{3}\ \pi\ \{r[l(t)]\}^2\ h[l(t)].$$
4. Relative error,
$$\epsilon = \frac{(\Delta v/\Delta t) - (dv/dt)}{(\Delta v/\Delta t)} \ .$$

E. Basic Procedure

1. Measure the diameter and calculate R.
2. Measure Q and calculate L.
3. Place tape along side of cone and choose b.
4. Determine $l^\star, h^\star$ and $r^\star$.
5. Fill the conical tank.
6. Drain the tank through an interval Δl of 2cm, recording the time interval Δt and the volume Δv captured in the measuring beaker. Repeat three times.
7. Drain the tank through an interval Δl of 1cm, recording the time interval Δt and the volume Δv captured in the measuring beaker. Repeat three times.
8. Calculate $\Delta v/\Delta t$ for each experiment.
9. Find $h(t)$ in terms of $l(t)$.
10. Find $r(t)$ in terms of $l(t)$.
11. Find $v(t)$ in terms of $l(t)$.
12. Find an expression for dv/dt in terms of $l(t)$.
13. Evaluate dv/dt for each experiment.
14. Evaluate ϵ for each experiment.
15. Plot ϵ for each experiment.

F. Detailed procedure

1. Use the ruler to measure the diameter of the top of the funnel and calculate the radius R. Record R in table 1.
2. Measure Q. Given α= 0.505 radians, use (D.1) to calculate L. Record Q and L in table 1.
3. Stick tape along a line from A to the top of the cone. Select a point along this line to be your point b, at which location the drainage rate will be estimated.
4. Measure the distance along the side of the cone between A and b and calculate $l^\star$, $h^\star$ and $r^\star$.
5. On the tape, mark a 2cm interval Δl centered at b (from 1cm above b to 1cm below b). Fill the funnel with water until the water level is approximately 2cm above b.
6. When the water level reaches the top mark (1cm above b), slide the graduated beaker under the outlet pipe and start the stopwatch. When the water level reaches the lower mark (1cm below b), remove the beaker from under the outlet pipe and stop the stopwatch. Record Δt in table 2. Measure the volume collected, Δv using the graduated cylinder and record it in table 2. Repeat the process three times.
7. On the tape mark a 1cm interval Δl centered at b (from 0.5cm above b to 0.5cm below b). Refill the funnel to approximately 2cm above b and allow to drain. When the water level reaches the top mark (0.5cm above b), slide the graduated beaker under the outlet pipe and start the stopwatch. When the water level reaches the lower mark (0.5cm below b), remove the beaker from under the outlet pipe and stop the stopwatch. Record Δt in table 2. Measure the volume collected, Δv using the graduated cylinder and record it in table 2. Repeat the process three time.
8. Use (D.2) to calculate $\Delta v/\Delta t$ for each experiment and record the value in column 2 of table 4.
9. Find an expression for $h(t)$ in terms of $l(t)$. Write it in table 3.
10. Find an expression for $r(t)$ in terms of $l(t)$. Write it in table 3.

11. Substitute your expressions (F.9) and (F.10) into (D.3) to obtain an expression for $v(t)$. Write it in table 3.
12. Differentiate your expression (F.10) using the chain rule and write it in table 3.
13. Evaluate dv/dt using $\Delta l/\Delta t$ to approximate dl/dt at $l^\star$ and record the value in column 3 of table 4.
14. Use (D.4) to evaluate ϵ for each experiment and record the values in column 4 of table 4.
15. Plot ϵ for each experiment on the graph provided.

Quantity	
R	
Q	
L	
$l^\star$	
$h^\star$	
$r^\star$	

Table 1: Dimensions of the conical tank

Trial	Δl	Δt	Δv
1	2cm		
2	2cm		
3	2cm		
4	1cm		
5	1cm		
6	1cm		

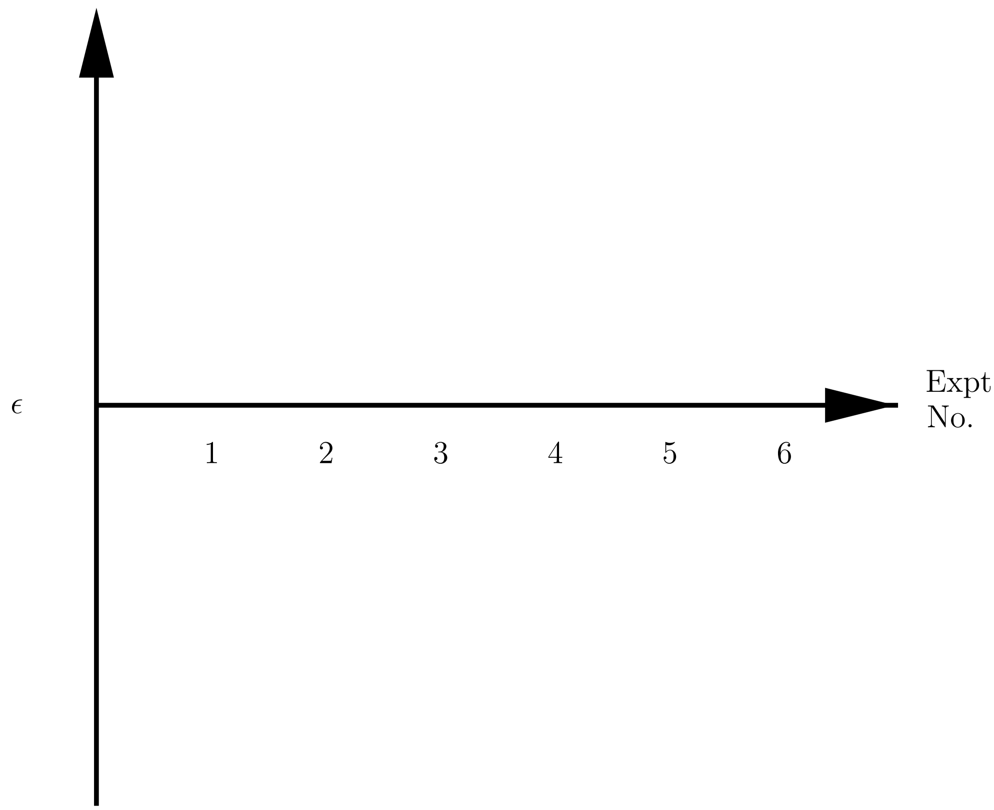
Table 2: Raw experimental data

Variable	Expression
$r(t)$	
$h(t)$	
$v(t)$	
dv/dt	

Table 3: Algebraic expressions

Trial	$\Delta v/\Delta t$	dv/dt	ϵ
1			
2			
3			
4			
5			
6			

Table 4: Measured and calculated values of the drainage rate and their relative error



ϵ
Expt.
No.
1
2
3
4
5
6

Discussion

1. What is the average relative error when $\Delta l = 2$cm.

2. What is the average relative error when $\Delta l = 1$cm.

3. Suggest reasons why you would expect to see a difference in the relative errors between these two sets of experiments.

4. Speculate as to why you do not see that difference (we would like more than a simple list of sources of experimental error). *Hint:* imagine a graph of $l(t)$ versus t. What would this graph have to look like for there to be a significant difference in error?