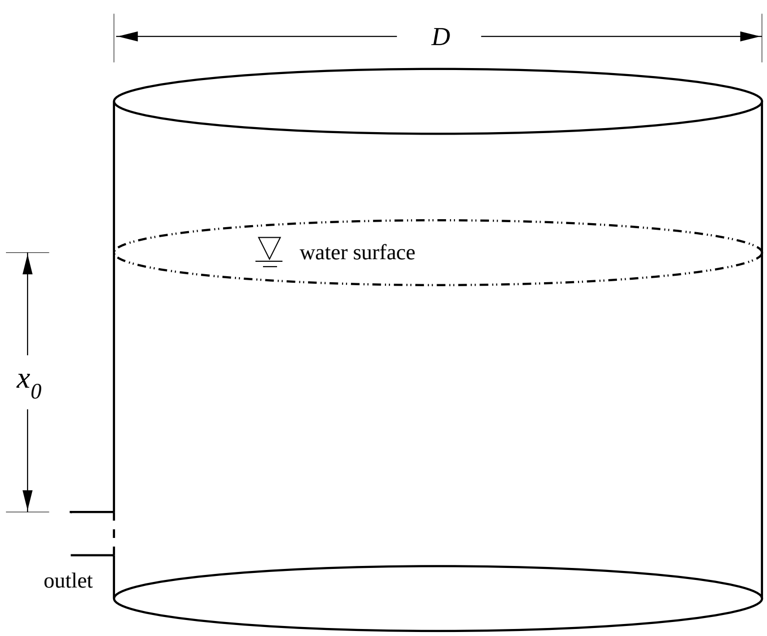
Understanding ϵ and δ; aka limits and derivatives

`Purpose:` This experiment provides an application of the concept of ϵ and δ that appears in the formal definition of a limit and shows the relationship between the notion of a limit and the notion of a derivative.

A. Apparatus

1. circular tank
2. water
3. stopwatch
4. ruler
5. container
6. masking tape
7. permanent marker
8. paper towels
9. bowl (stand)

B. Diagram



C. Nomenclature

1. D = interior diameter of the tank.
2. R = interior radius of the tank.
3. $n+1$ = the number of measurements of the depth of water in the tank.
4. t_0 = the initial time.
5. x_0 = the initial depth of the water in the tank measured from the top of the outlet.
6. v_0 = the initial volume of water in the tank.
7. t_i = time of the ith depth measurement, $i = 1, \ldots n$.
8. x_i = depth of the water at time t_i, $i = 1, \ldots n$.
9. v_i = volume of water at time t_i, $i = 1, \ldots n$.
10. y_i = the average drainage rate between t_{i-1} and t_i.
11. $y^\star$ = the target drainage rate.
12. $x^\star$ = the depth corresponding to $y^\star$.
13. x_l = lower limit of the range of depths guaranteed to contain $x^\star$.
14. x_u = upper limit of the range of depths guaranteed to contain $x^\star$.
15. ϵ = tolerance in the drainage rate.
16. δ = distance from $x^\star$ required to guarantee tolerance ϵ about $y^\star$.

D. Equations

1. $D = 2R$
2. Volume of water in the tank at time t_i,

$$v_i = \pi R^2 x_i, \; i = 0, \ldots, n.$$

3. Average drainage rate during the nth time interval,

$$y_i = -\left(\frac{v_i - v_{i-1}}{t_i - t_{i-1}}\right), \; i = 1, \ldots, n.$$

E. Basic Procedure

1. Measure D.
2. Tape the tank and pick x_0.
3. Fill the tank.
4. Drain tank, recording $x_i,\ i = 0, \ldots, n$.
5. Determine $v_i,\ i = 0, \ldots, n$.
6. Determine $y_i,\ i = 0, \ldots, n$.
7. Graph the results.
8. Select $y^\star$.
9. Determine x_l and x_u.
10. Record δ and ϵ.
11. Estimate $x^\star$.

F. Detailed Procedure

1. Use the ruler to measure D and calculate R using (D.1). Record R in table 2.

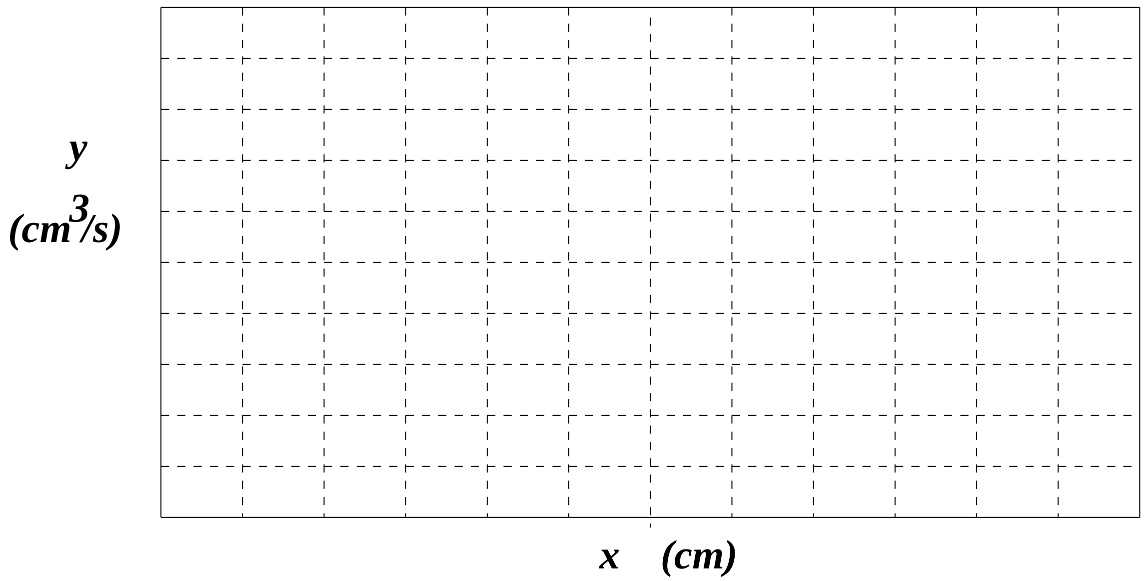
   `(Practice makes perfect.)` *Perform steps 2–4 twice. Select the better set of data for the subsequent analysis.*

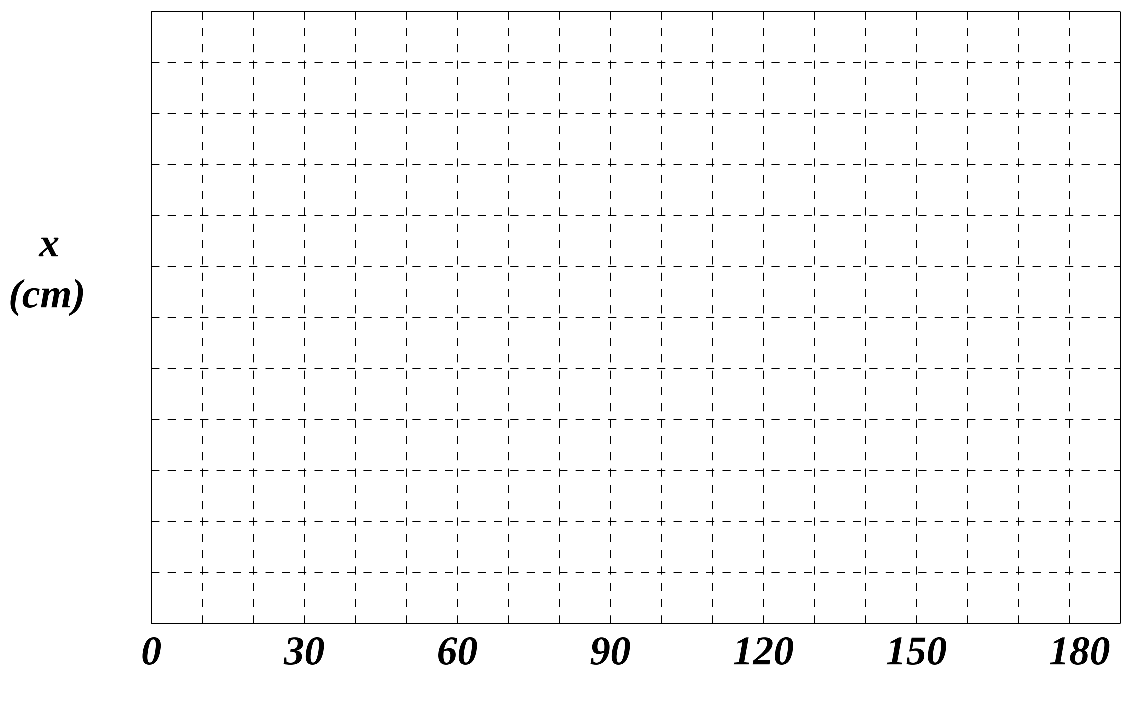
2. Place a strip of tape vertically up one side of the tank. Mark a point on the tape near the top of the tank and label it x_0.
3. Close the hose clamp and fill the tank to a depth larger than x_0.
4. Assign one group member to hold the apparatus on the stand and to keep the stand from floating. Open the hose clamp fully, allowing the water to drain continuously. Start the stopwatch when the water level corresponds to x_0 and then mark the level of the water surface on the tape every 10 seconds until the water level is just above the location of the outlet. Measure x_0 through x_n from the marks on the tape. Be consistent: measure between either the tops or the bottoms of the lines. Record $x_0, \ldots, x_n$ in table 1.
5. Use (D.2) to determine $v_i,\ i = 0, \ldots n$ and record in table 1.

6. Use (D.3) to determine $y_i, \ i = 1, \ldots n$ and record in table 1. Use your judgement with regard to how many significant digits to record.
7. Graph $x_i, \ i = 0, \ldots n$ as a function of time on the graph paper provided. Graph $y_i, \ i = 1, \ldots n$ as a function of depth on the graph paper provided.
8. Choose any target drainage rate, $y^\star$, and record $y^\star$ in table 2.
9. If possible, determine x_l and x_u, the range of depths within which the drainage rate may be equal to $y^\star \pm 2$ cm^3/s. Use your graph. Record x_l and x_u in table 2.
10. Determine the values for ϵ and δ corresponding to this experiment and record them in table 2.
11. Estimate $x^\star$ and record it in table 2.

n	t_i (sec)	x_i (cm)	v_i (cm^3)	y_i (cm^3/sec)
0	$t_0 = 0$	$x_0 =$	$v_0 =$	
1	$t_1 = 10$	$x_1 =$	$v_1 =$	$y_1 =$
2	$t_2 = 20$	$x_2 =$	$v_2 =$	$y_2 =$
3	$t_3 = 30$	$x_3 =$	$v_3 =$	$y_3 =$
4	$t_4 = 40$	$x_4 =$	$v_4 =$	$y_4 =$
5	$t_5 = 50$	$x_5 =$	$v_5 =$	$y_5 =$
6	$t_6 = 60$	$x_6 =$	$v_6 =$	$y_6 =$
7	$t_7 = 70$	$x_7 =$	$v_7 =$	$y_7 =$
8	$t_8 = 80$	$x_8 =$	$v_8 =$	$y_8 =$
9	$t_9 = 90$	$x_9 =$	$v_9 =$	$y_9 =$
10	$t_{10} = 100$	$x_{10} =$	$v_{10} =$	$y_{10} =$
11	$t_{11} = 110$	$x_{11} =$	$v_{11} =$	$y_{11} =$
12	$t_{12} = 120$	$x_{12} =$	$v_{12} =$	$y_{12} =$
13	$t_{13} = 130$	$x_{13} =$	$v_{13} =$	$y_{13} =$
14	$t_{14} = 140$	$x_{14} =$	$v_{14} =$	$y_{14} =$
15	$t_{15} = 150$	$x_{15} =$	$v_{15} =$	$y_{15} =$
16	$t_{16} = 160$	$x_{16} =$	$v_{16} =$	$y_{16} =$

Table 1: Data for 10-second intervals


y
(cm3/s)
x (cm)


x
(cm)
0
30
60
90
120
150
180
t (s)

R	
$y^\star$	
x_u	
x_l	
ϵ	
δ	
$x^\star$	

Table 2: Results

G. Discussion

1. (Paragraph) Write the $\epsilon - \delta$ definition for a limit in your own words, in terms of this experiment. The terms "flow rate" and "depth" should appear in your answer.

2. How is the graph of y vs x related to the graph of x vs t?

   (*Hint:* how is the point y_i on the graph of y vs x related to the points x_i and x_{i-1} on the graph of x vs t?)

3. Consider (D.3). Replace x_i and x_{i-1} by x_u and x_l, and replace $(t_i - t_{i-1})$ by Δt, the time interval it takes for the water surface to pass from x_u to x_l. Now consider the experiment as Δt gets smaller.

   (a) How do x_u and x_l change relative to each other?

   (b) Rewrite (D.3) to express the instantaneous flow rate as Δt gets smaller and smaller.

   (c) Express the instantaneous flow rate in terms of the time rate of change of the water depth.

4. *Extra Credit:* Explain why the graph of x vs t is relatively smooth, while the graph of y vs x is pretty rough, despite the fact that the data for both graphs comes from the same experiment.