

**$O(2)$ -symmetry breaking bifurcation:  
with application to the flow past a sphere in  
a pipe**

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## Abstract

The steady, axisymmetric laminar flow of a Newtonian fluid past a centrally-located sphere in a pipe first loses stability with increasing flow rate at a steady,  $O(2)$ -symmetry breaking bifurcation point. Using group theoretic results, a number of authors have suggested techniques for locating singularities in branches of solutions which are invariant with respect to the symmetries of an arbitrary group. We present these arguments for the  $O(2)$  symmetry encountered here and discuss their implementation for  $O(2)$ -symmetric problems. In particular, we describe how a bifurcation point may first be detected and then accurately located using an “extended system”. Also it is shown how to decide numerically if the bifurcating branch is sub-critical or super-critical. The numerical solutions were obtained using the finite-element code ENTWIFE. This has enabled us to compute the symmetry-breaking bifurcation point for a range of sphere to pipe diameter ratios. We also introduce a wire along the centreline of the pipe downstream of the sphere, and show its effect on the critical Reynolds number to be small.

**Keywords:** Flow past a sphere, Numerical bifurcation,  $O(2)$ -symmetry breaking.

# 1 Introduction

The external flow past a sphere has attracted considerable attention for at least the past 60 years since the pioneering experimental work of Möller [1]. More recent experimental investigations by Goldberg & Florsheim [2], Margarvey & Bishop [3], Nakamura [4] and Wu & Faeth [5], and computational studies by Natarajan & Acrivos [6], Tomboulides et al. [7] and Tavener [8] have all concluded that the initially steady axisymmetric flow past the sphere loses stability to a steady, asymmetric flow above a critical flow rate. The experimental studies and the computations of Tomboulides et al. [7] have shown that this steady asymmetric flow in turn loses stability to a time-dependent flow with a further small increase in the flow rate.

Natarajan & Acrivos [6] and Tavener [8] both used the finite-element method to compute the steady axisymmetric flow past a sphere. Natarajan & Acrivos [6] investigated the classical exterior problem in an unbounded domain, while Tavener [8] located the sphere along the axis of a pipe, and thereby introduced a new parameter, the “blockage ratio”, the ratio of sphere diameter to pipe diameter. The basic axisymmetric flow is linearly stable if and only if all eigenvalues of a large generalized non-symmetric eigenvalue problem lie in the stable half plane, but the flow loses stability as soon as a single eigenvalue or eigenvalue pair crosses the imaginary axis into the unstable half plane. Natarajan & Acrivos [6] employed a shift and invert strategy on a complex matrix eigenvalue problem, while Tavener [8] used a Cayley Transform technique [9] on an equivalent real matrix eigenvalue problem of the same dimension. In both the unbounded domain and in the pipe for blockage ratios between 0.2 and 0.7, a real eigenvalue corresponding to an eigenvector with an azimuthal wavenumber of 1, was determined to be the first to cross the imaginary axis, indicating the loss of stability to a steady asymmetric flow with azimuthal wavenumber one.

Relatively few numerical studies have investigated the role played by the nonslip lateral boundaries which obtain when the sphere is located in a tube. Johansson [10] computed the axisymmetric flow past a sphere in a cylinder at a single blockage ratio of 0.1, and more recently Wham et al [11] have calculated the flow for a range of blockage ratios. Bozzi et al. [12] have examined the much more difficult problem of flow past deformable drops in tubes. The major focus of all these studies has been the drag on the sphere (and deformations for [12]) rather than the stability of the computed axisymmetric flows.

In this paper we show how results from group theory applied to bifurcation problems can be used to reduce the complex matrix eigenvalue problem solved by Natarajan & Acrivos [6] to the real matrix eigenvalue problem investigated by Tavener [8]. We also show how an appropriately defined “extended system” [13] may be used to calculate the symmetry-breaking bifurcation points detected by both authors in a more robust and efficient manner. However, eigenvalue calculations are still required to ensure that the calculated bifurcation point corresponds to the instability which occurs for the lowest flow rate. It is important to note that even when there is a zero real eigenvalue with a corresponding singular Jacobian matrix, great care is needed when trying to locate this zero eigenvalue. The standard “change in sign of the determinant” technique can be made to work only if an appropriately restricted Jacobian is used.

The importance of symmetry in bifurcation problems is well-known, and many of the key concepts, including the connection with group representation theory, were introduced or clarified in the books by Vanderbauwhede [14], Golubitsky and Schaeffer [15] and Golubitsky, Stewart and Schaeffer [16]. The systematic numerical analysis of bifurcation in the presence of symmetry was started by Brezzi et al. [17], Werner & Spence [13] and Werner [18] for simple groups (eg.  $Z_2$ ). Dellnitz & Werner [19] and Werner [20] continued the detailed study for more general finite groups (eg.  $D_n$ ). Independently, Healey and his students [21], [22], [23], [24], motivated by examples in structural mechanics, and Aston [25], [26], motivated by a problem in water waves, developed similar theories but which allowed for infinite groups. An important idea is that, under the action of a group and for an appropriately chosen basis, the Jacobian of a nonlinear problem “block-diagonalizes”, with consequent savings for numerical algorithms, and improved theoretical understanding [20], [23], [26]. Many of the group theoretic ideas can appear complicated, and one aim of this paper is to explain them as simply as possible for the  $O(2)$  group, which is relevant for many problems defined on a cylindrical domain.

It is important to note that the results of this paper apply to all  $O(2)$ -symmetric problems and not just the example studied by Natarajan & Acrivos [6] and Tavener [8]. Group theoretic methods provide a rigorous justification of the considerable saving in computational expense when solving the linearized stability problem for all physical problems involving the breaking of an  $O(2)$  symmetry. Similar savings in computational effort may be sought by the application of general group theoretic constructs to other more exotic

dihedral or orthogonal symmetries (see, for example, Dellnitz and Werner [19] and Dellnitz [27]). Here we simply illustrate their application to the familiar and common  $O(2)$  symmetry group and demonstrate the advantages that can be accrued when studying the problem of flow past a sphere. More generally, once an underlying group theoretic structure is appreciated, very similar extended systems may be defined for the breaking of other symmetries. For example, Tavener et al. [28] efficiently compute the breaking of two distinct types of rotational invariances using this approach in their study of electro-hydrodynamic convection in nematic liquid crystals.

The detailed plan of the paper is as follows. In section 2, we describe the general bifurcation theory for the particular example of  $O(2)$  symmetry, and explain in detail the technique for detecting both steady and Hopf bifurcations. In particular we make the connection between the “appropriately restricted Jacobian” and the “block-diagonalisation” in the previous paragraphs. In addition we describe how a steady bifurcation may be accurately and efficiently computed using a generalization of the extended system presented in Werner and Spence [13], and how to decide if a bifurcating branch is sub- or super-critical. In section 3 we apply these results to the Navier-Stokes equations and boundary conditions for the problem of flow past a sphere in a pipe, explaining in detail the equivariance condition and writing out explicitly the important components in the  $O(2)$  theory. Section 4 contains a detailed discussion of the implementation of the extended system used to compute the bifurcation point in the physical example. In section 5 we discuss the implications of the theory in section 2 for the usual stability analysis, and hence give an improved stability analysis for the fluids problem which produces a real matrix eigenvalue problem directly. Numerical results are presented in section 6, where the critical Reynolds number is plotted as a function of blockage ratio. A minor change in the boundary conditions allows the effect of a supporting wire, or sting, to be determined.

## 2 Bifurcation theory in the presence of $O(2)$ symmetry

### 2.1 Preliminaries

Consider time dependent nonlinear problems of the form

$$By_t + f(y, \lambda) = 0, \quad y(t) \in H, \lambda \in \mathbf{R} \quad (1)$$

where  $H$  is a Hilbert space with inner product  $\langle \cdot, \cdot \rangle$ ,  $f : H \times \mathbf{R} \mapsto H$ , is a smooth nonlinear operator that is equivariant with respect to  $O(2)$ ,  $B$  is a linear operator on  $H$  that is equivariant with respect to  $O(2)$ , and  $y_t = \frac{dy}{dt}$ . To be precise about the equivariance condition, recall that  $O(2)$  is the Lie group generated by rotations  $r_\alpha$ ,  $\alpha \in \mathbf{R}$  and a reflection  $s$ , satisfying, for any  $\alpha, \beta \in \mathbf{R}$ ,

$$\begin{aligned} r_{\alpha+2\pi} &= r_\alpha, & r_{\alpha+\beta} &= r_\alpha r_\beta = r_\beta r_\alpha, \\ s^2 &= r_0 = r_{2\pi} = \mathbf{1}, & sr_\alpha &= r_{-\alpha}s. \end{aligned} \quad (2)$$

(Here  $\mathbf{1}$  is the group identity.) An action of  $O(2)$  on  $H$  is a continuous mapping  $\rho : O(2) \times H \mapsto H$ ,  $(\gamma, y) \mapsto \rho(\gamma, y) \equiv \gamma y$  such that  $\mathbf{1}y = y$  and  $(\gamma_1\gamma_2)y = \gamma_1(\gamma_2 y)$  for all  $y \in H$ ,  $\gamma_1, \gamma_2 \in O(2)$ . For any action  $\rho$  of  $O(2)$  on  $H$  we can define an orthogonal representation  $T$  in the space of linear homeomorphisms on  $H$

$$T(\gamma)y \equiv \gamma y,$$

through the choice of an  $O(2)$ -invariant inner product,

$$\langle T(\gamma)y_1, T(\gamma)y_2 \rangle = \langle y_1, y_2 \rangle \quad \forall y_1, y_2 \in H, \forall \gamma \in O(2). \quad (3)$$

To say that  $B$  and  $f$  are equivariant with respect to the action of  $O(2)$  means that

$$T(\gamma)B = BT(\gamma) \quad \forall \gamma \in O(2) \quad (4)$$

and

$$T(\gamma)f(y, \lambda) = f(T(\gamma)y, \lambda) \quad \forall \gamma \in O(2), \forall y \in H \quad (5)$$

where  $T(\gamma)$  is a representation of  $O(2)$ . An immediate consequence of (4) and (5) is that if  $y$  solves (1) for a given  $\lambda$ , then so does  $T(\gamma)y$  for all  $\gamma \in O(2)$ , giving a conjugacy class of solutions of (1).

A representation of  $O(2)$  on a Hilbert space appropriate for the analysis of the Navier-Stokes equations in cylindrical coordinates is given in §3. A much simpler example is the linear equation

$$-u'' + \lambda u = 0 \tag{6}$$

with  $2\pi$ -periodic boundary conditions over  $0 \leq x \leq 2\pi$ . It is easy to see that this equation is equivariant with respect to the representation of  $O(2)$  given by  $T(r_\alpha)u(x) = u(x + \alpha)$  and  $T(s)u(x) = -u(-x)$ . This is the natural representation for (6) and is used by Aston [25] in the analysis and numerical solution of the Kuramoto-Sivashinsky equation [29]. Other examples involving the group  $O(2)$  are found in Vanderbauwhede [14].

We are mainly interested in bifurcations from paths of steady  $O(2)$ -symmetric solutions of (1). Denote the set of  $O(2)$ -symmetric elements of  $H$  by  $H^{O(2)}$ , i.e.

$$H^{O(2)} := \{y \in H, y = T(\gamma)y, \forall \gamma \in O(2)\}.$$

An immediate consequence of (5) is that  $H^{O(2)}$  is invariant under  $f$  since, for  $y_{O(2)} \in H^{O(2)}$ ,

$$f(y_{O(2)}, \lambda) = f(T(\gamma)y_{O(2)}, \lambda) = T(\gamma)f(y_{O(2)}, \lambda) \tag{7}$$

and we may define  $f_{O(2)}$ , the restriction of  $f$  to  $H^{O(2)}$ . Hence, steady  $O(2)$ -symmetric solutions of (1) and paths of such solutions, say,

$$\mathcal{C} = \{(y(\lambda), \lambda) : y(\lambda) \in H^{O(2)}, \lambda \in I\}$$

where  $I$  is a given interval, may be computed by solving the reduced problem

$$f_{O(2)}(y_{O(2)}(\lambda), \lambda) = 0, \quad y_{O(2)} \in H^{O(2)}. \tag{8}$$

In §3 a problem defined on a cylindrical domain is considered, and in this case the  $O(2)$ -symmetric solutions are axisymmetric, and have zero swirl.

Define  $A(\lambda)$  to be the Jacobian matrix of  $f$  with respect to  $y$  along the path  $\mathcal{C}$ , i.e.

$$A(\lambda) := f_y(y(\lambda), \lambda), \quad (y(\lambda), \lambda) \in \mathcal{C}. \tag{9}$$

By differentiating (5) with respect to  $y$  we have

$$T(\gamma)f_y(y(\lambda), \lambda) = f_y(T(\gamma)y(\lambda), \lambda)T(\gamma)$$

and if  $y(\lambda) \in H^{O(2)}$ , then  $T(\gamma)y(\lambda) = y(\lambda)$ , producing

$$T(\gamma)A(\lambda) = A(\lambda)T(\gamma), \quad (10)$$

which is the same equivariance condition as that satisfied by  $B$  given by equation (4).

It is a standard result (see for example Aston [26] Theorems 2.6 and 2.11) that for the special case of the group  $O(2)$ , there is a unique orthogonal decomposition of  $H$

$$H = \sum_{m=0}^{\infty} \oplus V_m, \quad V_m \perp V_l, \quad m \neq l, \quad (11)$$

where  $V_m$  is an *isotypic* component of  $H$ , ie. an  $O(2)$ -invariant subspace of  $H$ , with the additional property in this case that  $V_m$  is irreducible. In other words, for  $y \in V_m$ , we have  $T(\gamma)y \in V_m$  for all  $\gamma \in O(2)$  and  $V_m$  has no proper  $O(2)$ -invariant subspaces.

It is important to note that for a general group, the isotypic components are usually *not* irreducible, as can be seen in Aston [25] where an example involving the dihedral group  $D_4$  is discussed in detail. However the group  $O(2)$ , though infinite, has a rather simple structure and perhaps for this reason is not often studied in detail in bifurcation texts. An exception to this is the early book by Vanderbauwhede [14] which contains many interesting and informative examples with  $O(2)$  symmetry.

Equation (11) may be compared to the well-known decomposition of  $H$  under the action of  $Z_2$ , the group  $\{1, s\}$ , with  $s$  a reflection. Then  $H$  can be decomposed into *symmetric* and *antisymmetric* subspaces

$$H = H^s \oplus H^a \quad (12)$$

where  $y \in H^s \Rightarrow sy = y$ , and  $y \in H^a \Rightarrow sy = -y$ .

For the  $2\pi$ -periodic example (6) mentioned in §2.1, the isotypic components are easily shown to be  $V_0 = \text{span}\{1\}$ , i.e. the constant functions, and

$$V_m = \text{span}\{\cos mx, \sin mx\}, \quad m = 1, 2, \dots, \quad (13)$$

as one would expect from the theory of Fourier series. In fact, for the group  $O(2)$ , equation (11) merely says that any element of  $H$  can be expressed uniquely as a Fourier series in  $\cos mx$  and  $\sin mx$ . In this case, the invariance of  $V_m$  is easily shown, since if  $u(x) = \cos mx$ , then

$$T(r_\alpha) \cos mx = \cos m\alpha \cos mx - \sin m\alpha \sin mx, \quad (14)$$

and if  $u(x) = \sin mx$  then

$$T(r_\alpha) \sin mx = \sin m\alpha \cos mx + \cos m\alpha \sin mx. \quad (15)$$

## 2.2 Eigenvalue theory for $O(2)$ -equivariant linear operators

Standard linear stability arguments show that a linearly stable  $O(2)$ -symmetric solution will lose stability at  $\lambda = \lambda_0$  when the generalized eigenvalue problem

$$\mu B\phi = A(\lambda)\phi \quad (16)$$

where  $B$  and  $A(\lambda)$  are linear operators on  $H$ , has an eigenvalue  $\mu(\lambda_0)$  with  $Re(\mu(\lambda_0)) = 0$  and  $\frac{d}{d\lambda} [Re(\mu(\lambda_0))] \neq 0$ . If  $\mu(\lambda_0)$  is real then there is a steady bifurcation and if  $\mu(\lambda_0)$  is complex then there is a Hopf bifurcation, but care is needed since the presence of  $O(2)$  symmetry means that most of the eigenvalues of (16) occur with multiplicity two. However, we shall show that since  $B$  and  $A(\lambda)$  are  $O(2)$ -equivariant, the problem (16) may be reduced to one in which simple eigenvalues of a real generalized eigenvalue problem are sought.

In [20], Werner provides a complete block diagonalization theory for the eigenvalue problem  $A(\lambda)\phi = \mu\phi$  where  $A(\lambda)$  is an  $n \times n$  real matrix which satisfies an equivariance condition with respect to a compact Lie group of  $n \times n$  orthogonal matrices. Aston, [26], discusses equivariant bifurcation theory for general compact Lie groups, including  $O(2)$ , applied to problems defined on a Hilbert space. Healey & Treacy [23] present similar results for structures problems, and these were used by Wohlever & Healey [24] in an analysis of an axially compressed cylindrical shell. In this subsection we use the results in these three papers to give the theory for a generalized eigenvalue problem  $\mu B\phi = A(\lambda)\phi$  where  $B$  and  $A(\lambda)$  are  $O(2)$ -equivariant linear operators on a Hilbert space.

The following theorem is a key result.

**Theorem 2.2.1** *Let  $A$  be any  $O(2)$ -equivariant linear operator on the Hilbert space  $H$ . Then  $A : V_m \rightarrow V_m$  where the  $V_m$  are as in (11).*

**PROOF:** This result is a special case of Theorem 3.1 in Aston [26] (see also §2.3 of [20]). However, for the group  $O(2)$  a straightforward proof by contradiction can be given. Assume that for some  $y_m \in V_m$ ,  $Ay_m = z_m + z_n$ ,

with  $z_m \in V_m$  and  $z_n \in V_n, m \neq n$ . Now with  $T(\gamma) = T(r_\alpha)$ , for some  $\alpha \in [0, 2\pi)$ , and using (14) and (15) it is seen that  $T(r_\alpha)y_m$  is a linear combination of  $\cos m\alpha$  and  $\sin m\alpha$ . Thus  $AT(r_\alpha)y_m$  is also a linear combination of these terms. However  $T(r_\alpha)Ay_m = (T(r_\alpha)z_m + T(r_\alpha)z_n)$  will be a linear combination of  $\cos m\alpha, \sin m\alpha, \cos n\alpha$ , and  $\sin n\alpha$ . This contradicts the identity  $AT(r_\alpha) = T(r_\alpha)A, \forall \alpha$ , unless  $z_n = 0$ . Clearly this argument extends to cover the assumption that  $Ay_m = z_m + z_m^\perp, z_m^\perp \in V_m^\perp$ .  $\square$

It is easily shown using (3) and (4) that if  $A$  is  $O(2)$ -equivariant then so is the adjoint operator  $A^*$ . Thus we have the following Corollary.

**Corollary 2.2.1** *Let  $A$  be any  $O(2)$ -equivariant linear operator on the Hilbert space  $H$ . Then  $A^* : V_m \rightarrow V_m$  where  $A^*$  is the adjoint of  $A$  and the  $V_m$  are as in (11).*

Since  $B$  and  $A(\lambda)$  are both  $O(2)$ -equivariant, Theorem 2.2.1 shows that the eigenvalue problem  $\mu B\phi = A(\lambda)\phi$ , see (16), decouples into the (infinite) set of finite-dimensional generalized eigenvalue problems

$$\mu B_m \phi = A_m(\lambda)\phi, \quad \phi \in V_m, \quad m = 0, 1, 2, \dots \quad (17)$$

where  $B_m$  and  $A_m(\lambda)$  are the restrictions of  $B$  and  $A(\lambda)$  to  $V_m$  respectively. Equivalently, using the notation of [26]

$$\mu \operatorname{diag}(B_m)\phi = \operatorname{diag}(A_m(\lambda))\phi, \quad \phi \in H, \quad m = 0, 1, 2, \dots \quad (18)$$

and it is this decomposition which leads to the use of the term “block-diagonalization”, even when talking about infinite-dimensional operators, see for example Healey & Treacy [23].

Roughly speaking, one can think of the isotypic decomposition (11) as being produced by the “rotation” elements  $r_\alpha$  of  $O(2)$ . The fact that there is also a “reflection” element has not yet been used, and indeed Theorem 3.3, Aston [26] applied to  $O(2)$ , shows there is a finer block decomposition

$$V_m = V_m^s \oplus V_m^a, \quad m = 1, 2, \dots \quad (19)$$

where  $V_m^s$  and  $V_m^a$  are the symmetric and antisymmetric components of  $V_m$ , see (12). Further, (Theorem 2.3 [20], Theorem 3.3 [26], and Figure 3 of [24]),  $V_m^s$  and  $V_m^a$  are invariant subspaces of  $B$  and  $A(\lambda)$ , and if we denote the

restriction of  $A_m(\lambda)$  to  $V_m^s$  ( $V_m^a$ ) by  $A_m^s$  ( $A_m^a$ ) and similarly for  $B_m$ , then for  $m = 1, 2, \dots$ , (18) can be further reduced to

$$\mu \begin{pmatrix} B_m^s & 0 \\ 0 & B_m^a \end{pmatrix} \begin{pmatrix} \phi^s \\ \phi^a \end{pmatrix} = \begin{pmatrix} A_m^s & 0 \\ 0 & A_m^a \end{pmatrix} \begin{pmatrix} \phi^s \\ \phi^a \end{pmatrix}, \quad m = 1, 2, \dots \quad (20)$$

with crucially

$$B_m^s = B_m^a \quad \text{and} \quad A_m^s = A_m^a, \quad m = 1, 2, \dots \quad (21)$$

Thus  $A_m(\lambda)$ ,  $m = 1, 2, \dots$  is diagonalized into two identical sub-blocks, and similarly for  $B_m$ . For the linear problem (6), the two identical sub-blocks are  $0 = m^2 + \lambda$ ,  $m = 1, 2, \dots$

Returning to the eigenvalue problem  $\mu B\phi = A(\lambda)\phi$ , we have therefore proved the following lemma (see also Proposition 3.6 of [20]).

**Lemma 2.2.1** *Let  $A(\lambda)$  and  $B$  be  $O(2)$ -equivariant.*

(a)  *$(\mu, \phi)$  is an eigenpair of  $\mu B\phi = A(\lambda)\phi$  iff  $(\mu, \phi)$  is an eigenpair of  $\mu B_m\phi = A_m(\lambda)\phi$ ,  $\phi \in V_m$  for some  $m$ .*

(b) *If  $m \neq 0$  in (a), then  $\mu$  has geometric multiplicity 2, with the corresponding eigenvectors  $\phi^s$  and  $\phi^a$  satisfying  $\mu B_m^s \phi^s = A_m^s(\lambda)\phi^s$  and  $\mu B_m^a \phi^a = A_m^a(\lambda)\phi^a$ , with  $B_m^s = B_m^a$  and  $A_m^s(\lambda) = A_m^a(\lambda)$ .*

**Definition 2.2.1** *Following Definition 3.7 in [20], we call  $\mu$  an  $O(2)$ -simple eigenvalue if*

(a) *there is only one  $m$  for which  $\mu$  is an eigenvalue of  $\mu B_m\phi = A_m(\lambda)\phi$ , and*

(b) *if in (a),  $m = 0$ ,  $\mu$  is an algebraically simple eigenvalue of  $\mu B_0\phi = A_0(\lambda)\phi$ , or if  $m \neq 0$ ,  $\mu$  is an algebraically simple eigenvalue of  $\mu B_m^s\phi = A_m^s(\lambda)\phi$  (and  $\mu B_m^a\phi = A_m^a(\lambda)\phi$ ).*

It is now a simple matter (see §3.2 of [20]) to show that as the parameter  $\lambda$  which describes the curve  $\mathcal{C}$  varies,  $O(2)$ -simple eigenvalues of  $\mu B\phi = A(\lambda)\phi$  behave like algebraically simple eigenvalues of matrices with no equivariance properties.

## 2.3 Steady state bifurcation theory

The bifurcation theory presented here is mainly the restriction of the results of §3 of [26] to the  $O(2)$  case applied to (1).

Assume the curve of  $O(2)$ -symmetric solutions is being computed using (8). Further, assume that  $(y(\lambda_0), \lambda_0) \in \mathcal{C}$  and that  $\mu(\lambda_0) = 0$  is an  $O(2)$ -simple eigenvalue of  $\mu B\phi = A(\lambda_0)\phi$ . Let  $\mathcal{N}_0 = \text{Null}(A(\lambda_0))$ . Clearly  $\mathcal{N}_0 \subset V_m$  for some  $V_m$  from Lemma 2.2.1. If  $\mathcal{N}_0 \subset V_0$  i.e.  $\mathcal{N}_0 \cap H^{O(2)} \neq \{0\}$ , then even if bifurcation occurs, the  $O(2)$  symmetry will not be broken, hence this would be an  $O(2)$ -symmetry preserving bifurcation (generically, a turning point or fold point in the space of  $O(2)$ -symmetric functions). This case is of less interest to us and is excluded from the next theorem by the following assumption.

We assume

**Assumption 2.3.1**  $\mathcal{N}_0 \subset V_m$ , for some  $m \neq 0$ ,

and

**Assumption 2.3.2**  $\mu$  is an  $O(2)$ -simple eigenvalue i.e.

$$\dim(\mathcal{N}_0 \cap H^s) = 1 \quad \text{and} \quad \dim(\mathcal{N}_0 \cap H^a) = 1$$

where  $H^s = \{y : sy = y\}$  and  $H^a = \{y : sy = -y\}$ , the symmetric and antisymmetric elements of  $H$ .

Before stating the main bifurcation result we first need to introduce some terms. We define the scalar  $b_\lambda$  by

$$b_\lambda := \langle \psi_0, f_{y\lambda}(y(\lambda_0), \lambda_0)\phi_0 + f_{yy}(y(\lambda_0), \lambda_0)\phi_0 v \rangle \quad (22)$$

where  $\phi_0 \in V_m^s := V_m \cap H^s$ ,  $\psi_0 \in \text{Null}(f_y(y(\lambda_0), \lambda_0)^*) \cap H^s$ , where  $*$  denotes the adjoint operator, and  $v \in H^{O(2)}$  is the unique solution of

$$f_y(y(\lambda_0), \lambda_0)v + f_\lambda(y(\lambda_0), \lambda_0) = 0, \quad (23)$$

since  $f_\lambda(y(\lambda_0), \lambda_0) \in H^{O(2)}$  for  $y(\lambda_0) \in H^{O(2)}$ . In fact if  $v \in H^{O(2)}$ , and  $\phi_0 \in V_m^s$ , then  $f_{y\lambda}(y(\lambda_0), \lambda_0)\phi_0 \in V_m^s$ , and  $f_{yy}(y(\lambda_0), \lambda_0)\phi_0 v \in V_m^s$ . (These are proved after differentiating (5) appropriately.)

The next theorem provides the main bifurcation result. It is derived from Theorems 3.4 and 3.5 of [26] and Theorem 6.4.3 of [14].

**Theorem 2.3.1** *Let  $y(\lambda_0) \in H^{O(2)}$ . Suppose  $A(\lambda_0)$  is Fredholm of index zero and that Assumptions 2.3.1 and 2.3.2 hold. Assume the non-degeneracy condition*

$$b_\lambda \neq 0. \quad (24)$$

Then there exists a secondary branch of  $O(2)$ -symmetry breaking solutions. The bifurcation is of pitchfork type and the bifurcating branch has  $D_m$  symmetry, where  $m$  is as in Assumption 2.3.1. (Here  $D_m$  is the dihedral group generated by  $r_{2\pi/m}$  and  $s$ .)

PROOF: The existence of a branch follows from the Equivariant Branching Lemma [30], [14] (see also [26]). The fact that the bifurcation is pitchfork follows from case (ii) of Theorem 3.5 of [26] since  $\mathbf{N}_{O(2)}(Z_2)/Z_2$  is isomorphic to  $Z_2$ , where  $\mathbf{N}_{O(2)}(Z_2)$  is the normalizer of  $Z_2$  in  $O(2)$  defined by

$$\mathbf{N}_{O(2)}(Z_2) = \{\gamma \in O(2) : \gamma\sigma = \sigma\gamma \forall \sigma \in Z_2\}.$$

The fact that the bifurcating branch is in  $D_m$  is proved in Theorem 6.4.3 of [14].  $\square$

**A test for steady bifurcations.** Note that unless  $m = 0$ , a zero eigenvalue of  $\mu B_m \phi = A_m \phi$  along a path of  $O(2)$ -symmetric solutions cannot be detected by looking for a change in sign in  $\det A(\lambda)$  or even  $\det A_m(\lambda)$  since two eigenvalues pass through zero at  $\lambda = \lambda_0$ . However, if only *steady* bifurcations are sought then a test based on detecting a sign change in  $\det A_m^s$  (or in  $\det A_m^a$ ) would work since both these matrices have a simple eigenvalue at  $\lambda = \lambda_0$ .

Once a steady bifurcation has been located approximately, it may be calculated accurately using a generalization of the “extended system” introduced by Werner and Spence [13] to compute pitchfork bifurcation points. The method is exactly as described in Theorem 3.6 of [26], which we specialize here to the  $O(2)$  case.

**Theorem 2.3.2** *Let  $(y(\lambda_0), \lambda_0) \in H^{O(2)} \times \mathbb{R}$  and assume assumptions 2.3.1 and 2.3.2 hold with  $\mathcal{N}_0 \subset V_m$  for some  $m \neq 0$ . Then the extended system*

$$\begin{aligned} G(z) &= 0, \quad G : Z \mapsto Z & (25) \\ \text{where } G(z) &= \begin{pmatrix} f(y, \lambda) \\ f_y(y, \lambda)\phi \\ \langle l, \phi \rangle - 1 \end{pmatrix}, \quad y = \begin{pmatrix} y \\ \phi \\ \lambda \end{pmatrix} \in Z \end{aligned}$$

with  $Z \in H^{O(2)} \times V_m^s \times \mathbb{R}$ ,  $l \in V_m^s$ , has an isolated solution  $(y_0, \phi_0, \lambda_0)$  where  $\phi_0 \in V_m^s$  iff the non-degeneracy condition (24) holds.

Note that the theorem is true if  $V_m^s$  is replaced by  $V_m^a$  throughout. The system  $G(z) = 0$  is known as an “extended system”.

When answering questions on linearized stability of steady solutions of (1), it is often important to know if the bifurcating branch is sub-critical or super-critical. To do this we need to calculate certain quantities involving higher derivatives of  $f_y$ . First note that by differentiating (5) and setting  $y \in H^{O(2)}$  then  $\forall T(\gamma)$ ,

$$T(\gamma)f_{yy}(y, \lambda)uv = f_{yy}(T(\gamma)y, \lambda) T(\gamma)u T(\gamma)v = f_{yy}(y, \lambda) T(\gamma)u T(\gamma)v.$$

Similarly, for  $y \in H^{O(2)}$  and  $\forall T(\gamma)$ ,

$$T(\gamma)f_{yyy}(y, \lambda)uvw = f_{yyy}(y, \lambda) T(\gamma)u T(\gamma)v T(\gamma)w.$$

Thus with  $\phi_0 \in V_m^s$  (ie.  $\phi_0$  is a multiple of  $\cos mx$ )

$$f_{yy}(y, \lambda)\phi_0\phi_0 \in H^{O(2)} \oplus V_{2m}^s$$

since  $\cos^2 mx = (1 + \cos 2mx)/2$ , and

$$f_{yyy}(y, \lambda)\phi_0\phi_0\phi_0 =: e_m + e_{3m} \in V_m^s \oplus V_{3m}^s$$

since  $\cos^3 mx = (3 \cos mx + \cos 3mx)/4$ . We introduce  $z = z_0 + z_{2m} \in H^{O(2)} \oplus V_{2m}^s$  respectively, where  $z$  is the unique solution of

$$f_y(y(\lambda_0), \lambda_0)z + f_{yy}(y(\lambda_0), \lambda_0)\phi_0\phi_0 = 0. \quad (26)$$

To decide on sub- or super-criticality we need to calculate  $b_\lambda$  in (22) and  $d$  defined by (see Cliffe & Spence [31])

$$d := \langle \psi_0, f_{yyy}(y(\lambda_0), \lambda_0)\phi_0\phi_0\phi_0 + 3f_{yy}(y(\lambda_0), \lambda_0)\phi_0z \rangle \quad (27)$$

where  $\phi_0$  and  $\psi_0$  are as in (22), and  $z$  is given by (26). Now, using Corollary 2.2.1, it is easily seen that  $\psi_0$  is perpendicular to  $V_m^\perp$ . Thus  $d$  can be found from

$$d := \langle \psi_0, e_m + 3f_{yy}(y(\lambda_0), \lambda_0)\phi_0z_0 \rangle \quad (28)$$

with  $e_m$  and  $z_0$  defined above. Note that  $d \neq 0$  ensures that there is a *quadratic* pitchfork bifurcation, whereas if  $d = 0$  then there would be a *quartic*, or even more degenerate, pitchfork bifurcation (see Cliffe & Spence [31]). The following Lemma provides the test for sub- or super-criticality.

**Lemma 2.3.1** *Assume  $d/b_\lambda \neq 0$  where  $b_\lambda$  and  $d$  are given in (22) and (28) respectively. Then the bifurcation is supercritical if  $d/b_\lambda < 0$ , and subcritical if  $d/b_\lambda > 0$ .*

**PROOF:** In the proof of Theorem 2.3.1 it is shown that the pitchfork bifurcation is essentially of  $Z_2$ -symmetry-breaking type. In §3 of [31] it is shown by a Lyapunov-Schmidt reduction procedure that the reduced equation for a  $Z_2$ -symmetry-breaking bifurcation is

$$b_\lambda \lambda x + \frac{d}{6} x^3 + h.o.t = 0$$

and so if  $d/b_\lambda < 0$  this is (contact) equivalent to the normal form  $\lambda x - x^3 = 0$ , which has super-critical bifurcating branches. Similarly,  $d/b_\lambda > 0$  leads to sub-critical bifurcating branches.  $\square$

## 2.4 Time dependent bifurcation theory

In this section we shall merely discuss how to detect a Hopf bifurcation point on a path of  $O(2)$ -symmetric steady states of (1). For more details on the numerical analysis we refer the reader to [19] and [27], and to [16] for a more complete account of bifurcation theory for nonlinear dynamical systems in the presence of  $O(2)$ -symmetry.

We shall content ourselves here with noting that  $O(2)$ -symmetry breaking Hopf bifurcations can only occur at  $\lambda = \lambda_0$  if  $\mu B \phi = A(\lambda_0) \phi$  has a complex eigenvalue with  $\text{Re}(\mu(\lambda_0)) = 0$ . Lemma 2.2.1 applies here also and immediately we see that symmetry breaking Hopf bifurcation points at  $\lambda = \lambda_0$  are only possible if, for  $m \neq 0$ ,  $\mu B_m^s \phi^s = A_m^s(\lambda_0) \phi^s$  (or equivalently  $\mu B_m^a \phi^a = A_m^a(\lambda_0) \phi^a$ ) has a purely imaginary eigenvalue pair. Assuming (see [20], definition 4.12) that  $\mu(\lambda_0)$  is simple and  $\frac{d}{d\lambda} [\text{Re}(\mu(\lambda_0))] \neq 0$ , then we say that  $(y_0, \lambda_0)$  is an  $O(2)$ -symmetry breaking Hopf bifurcation. Symmetry *preserving* Hopf bifurcations correspond to complex eigenvalues in the  $m = 0$  block,  $\mu B_0 \phi = A_0(\lambda_0) \phi$ .

Since the bifurcation example of interest in §3 is a steady bifurcation, we do not consider this further here. We complete this section with a statement of a general means of detecting steady or Hopf bifurcations.

**A test to detect a steady or Hopf bifurcations.** If both steady and Hopf bifurcations are sought along a path of  $O(2)$ -symmetric steady solutions, then the eigenvalues nearest the imaginary axis need to be found

for the problems  $\mu B_0 \phi = A_0 \phi$ ,  $\phi \in H^{O(2)}$ , and  $\mu B_m^s \phi^s = A_m^s \phi^s$ ,  $m = 1, 2, \dots$ . In practice one would choose an upper limit for  $m$ , say  $m_{max}$  and check the eigenvalues of the eigenvalue problems for  $m = 0, 1, 2, \dots, m_{max}$ . For the numerical calculations in §5,  $m_{max}$  was taken to be 4.

An account of how this may be accomplished using a technique based on the Cayley Transform [9] is given in Tavener [8].

### 3 The fluids problem

We now apply the abstract  $O(2)$ -symmetric bifurcation theory developed in §2, to the flow of a Newtonian fluid past a centrally located sphere in a pipe. In the usual cylindrical coordinates, the steady Navier-Stokes equations are

$$\begin{aligned} & Re \left( \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} \right) \\ & + \frac{\partial p}{\partial r} \\ & - \left( \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u_r}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) = 0, \end{aligned} \quad (29)$$

$$\begin{aligned} & Re \left( \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} \right) \\ & + \frac{1}{r} \frac{\partial p}{\partial \theta} \\ & - \left( \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u_\theta}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right) = 0, \end{aligned} \quad (30)$$

$$\begin{aligned} & Re \left( \frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \right) \\ & + \frac{\partial p}{\partial z} \\ & - \left( \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u_z}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} \right) = 0, \end{aligned} \quad (31)$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0. \quad (32)$$

In order to non-dimensionalize the Navier-Stokes equations as above, the diameter of the pipe  $D$  was chosen as the length scale and the mean velocity

$$\bar{U} = \frac{4}{\pi D^2} \int_0^{D/2} 2\pi r u_z^* dr,$$

as the velocity scale, where  $u_z^*$  is the (dimensional) parabolic axial velocity that would exist in an unobstructed pipe. The Reynolds number  $Re = D\bar{U}/\nu$ , where  $\nu$  is the kinematic viscosity. We impose nonslip boundary conditions on the surface of the sphere and along the walls of the pipe, and assume that a fully developed parabolic velocity profile is attained sufficiently far upstream and downstream of the sphere.

Let  $y(t)$  be a solution of equations (29) – (32), i.e.

$$y(t) = \begin{pmatrix} u_r(r, \theta, z) \\ u_\theta(r, \theta, z) \\ u_z(r, \theta, z) \\ p(r, \theta, z) \end{pmatrix},$$

where  $y(t)$  lies in Hilbert space  $H = W^{1,2}(\Omega)^3 \times L^2(D)$ . Equations (29) – (32) may be combined to define a mapping

$$f(y, Re) : H \times \mathbb{R} \rightarrow H. \quad (33)$$

The bounded linear operators

$$R_\alpha \begin{pmatrix} u_r(r, \theta, z) \\ u_\theta(r, \theta, z) \\ u_z(r, \theta, z) \\ p(r, \theta, z) \end{pmatrix} = \begin{pmatrix} u_r(r, \theta + \alpha, z) \\ u_\theta(r, \theta + \alpha, z) \\ u_z(r, \theta + \alpha, z) \\ p(r, \theta + \alpha, z) \end{pmatrix}, \quad \alpha \in [0, 2\pi), \quad (34)$$

and

$$S \begin{pmatrix} u_r(r, \theta, z) \\ u_\theta(r, \theta, z) \\ u_z(r, \theta, z) \\ p(r, \theta, z) \end{pmatrix} = \begin{pmatrix} u_r(r, -\theta, z) \\ -u_\theta(r, -\theta, z) \\ u_z(r, -\theta, z) \\ p(r, -\theta, z) \end{pmatrix} \quad (35)$$

are representations of the symmetry group  $O(2)$  on the Hilbert space  $H$ . It is straightforward to show that the operator  $f$  defined by (33) is equivariant with respect to both  $R_\alpha$  and  $S$ , i.e.

$$R_\alpha f(y, Re) = f(R_\alpha y, Re), \quad (36)$$

and

$$S f(y, Re) = f(Sy, Re). \quad (37)$$

Applying the theory in §2.2, especially equations (11) and (13), (see also Aston [25]) the solution space  $H$  may be decomposed as an orthogonal direct sum of finite-dimensional  $O(2)$ -irreducible subspaces  $V_m$ ,

$$H = V_0 \oplus V_1 \oplus V_2 \oplus \cdots$$

where

$$V_m = \text{span} \left\{ \begin{pmatrix} u_r^m(r, z) \cos m\theta \\ u_\theta^m(r, z) \sin m\theta \\ u_z^m(r, z) \cos m\theta \\ p^m(r, z) \cos m\theta \end{pmatrix}, \begin{pmatrix} u_r^m(r, z) \sin m\theta \\ u_\theta^m(r, z) \cos m\theta \\ u_z^m(r, z) \sin m\theta \\ p^m(r, z) \sin m\theta \end{pmatrix} \right\}. \quad (38)$$

A (matrix) representation of  $O(2)$  on the irreducible subspace  $V_m$  is

$$R_\alpha^m = \begin{pmatrix} C & D \\ -D & C \end{pmatrix}$$

where

$$C = \begin{pmatrix} \cos m\alpha & 0 & 0 & 0 \\ 0 & \cos m\alpha & 0 & 0 \\ 0 & 0 & \cos m\alpha & 0 \\ 0 & 0 & 0 & \cos m\alpha \end{pmatrix},$$

$$D = \begin{pmatrix} \sin m\alpha & 0 & 0 & 0 \\ 0 & -\sin m\alpha & 0 & 0 \\ 0 & 0 & \sin m\alpha & 0 \\ 0 & 0 & 0 & \sin m\alpha \end{pmatrix},$$

and

$$S^m = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$$

where

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Notice that each isotypic subspace is  $2(\times 4)$  dimensional.

Due to the symmetrical nature of the physical domain and to the boundary conditions we impose, there are solutions of the flow problem that are  $O(2)$ -invariant. Let  $H^{O(2)}$  be the subspace of  $O(2)$ -invariant solutions. Following the theory in §2 we define

$$f^{O(2)} : H^{O(2)} \rightarrow H^{O(2)} \quad (39)$$

to be the restriction of  $f$  to  $H^{O(2)}$ .

In order to compute  $O(2)$ -invariant solutions we solved equations (29) – (32) in the domain  $\Omega$  shown in Figure 1, which is a radial slice through the physical domain. Non-slip velocity boundary conditions were imposed on the surface of the sphere and along the walls of the pipe. At the inflow boundary only the axial component of the velocity field is non-zero and  $u_z(r)$  is both axisymmetric and time independent. The Dirichlet boundary

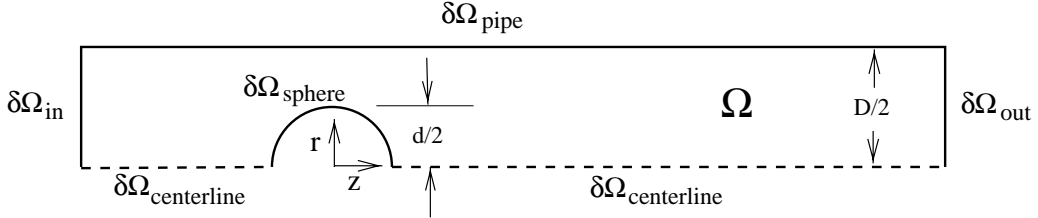
conditions applied were therefore

$$\begin{aligned}
u_r = u_\theta = u_z = 0 & \quad \text{on} \quad \partial\Omega_{\text{pipe}}, \\
u_r = u_\theta = u_z = 0 & \quad \text{on} \quad \partial\Omega_{\text{sphere}}, \\
u_r = u_\theta = 0, \quad u(z) = 2(0.5 - r)(0.5 + r) & \quad \text{on} \quad \partial\Omega_{\text{in}}. \quad (40)
\end{aligned}$$

The boundary conditions along the remaining boundaries of  $\Omega$  were chosen to be those arising as natural boundary conditions when a weak formulation of equations (29) – (32) is constructed, namely

$$\begin{aligned}
-p + \frac{\partial u_z}{\partial z} = 0, \quad \frac{\partial u_r}{\partial z} = \frac{\partial u_\theta}{\partial z} = 0 & \quad \text{on} \quad \partial\Omega_{\text{out}}, \\
-p + \frac{\partial u_r}{\partial r} = 0, \quad \frac{\partial u_\theta}{\partial r} = \frac{\partial u_z}{\partial r} = 0 & \quad \text{on} \quad \partial\Omega_{\text{centerline}}. \quad (41)
\end{aligned}$$

Figure 1



*Sketch of the physical domain.*

As a consequence of (36) and (37), for  $y \in H^{O(2)}$ , we can use equations (17) and (18) to decompose the Fréchet derivative  $f_y(y, Re)$  into block diagonal form

$$f_y(y, Re) = \text{diag}(f_y^m(y, Re)), \quad m = 0, 1, 2 \dots$$

where  $f_y^m = f_y|_{V_m}$  and  $f_y^m : V_m \rightarrow V_m$ .

Using (19), (20), and (21) we can further decompose the block diagonal structure of  $f_y$  for  $m \neq 0$ . To avoid a double subscript notation we define  $g_y^m : V_m^s \mapsto V_m^s$  and  $h_y^m : V_m^a \mapsto V_m^a$ , such that

$$f_y^m = \text{diag}(g_y^m, h_y^m), \quad m \neq 0,$$

where  $g_y^m$  and  $h_y^m$  are identical.

In this way we reduce the problem of finding a singular point of  $f_y$  to that of finding a singular point of  $g_y^m$  for some  $m$ . If  $g_y^m$  is singular for any  $m \neq 0$ , then Theorem 2.3.1 states that there will be a bifurcating branch of  $D_m$ -invariant solutions i.e. solutions that are invariant with respect to  $S$  and  $R_{2\pi/m}$ , and that the symmetry-breaking will occur at a pitchfork bifurcation point, which may be calculated using the extended system (25).

## 4 Solution procedure

We construct the weak form of equations (29) – (32) and boundary conditions (40) and (41), and discretize the  $O(2)$ -symmetric subspace  $H^{O(2)}$  using a finite-element approximation in a radial slice through the full cylindrical domain. The finite-dimensional approximating subspace  $\mathbf{H}^{O(2)} \subset H^{O(2)}$  of dimension  $N$ , say, was constructed using quadrilateral elements with bi-quadratic velocity interpolation and discontinuous linear pressure interpolation using the finite element code ENTWIFE [32]. Let  $\mathbf{f} : \mathbf{H}^{O(2)} \rightarrow \mathbf{H}^{O(2)}$  be the discretized weak form of the restriction of  $f$  to  $H^{O(2)}$ . Further, let  $\mathbf{V}_m^s$  and  $\mathbf{V}_m^a$  be the discretizations versions of subspaces  $V_m^s$  and  $V_m^a$  respectively and  $\mathbf{g}_y^m$  and  $\mathbf{h}_y^m$  be the discretized versions of Fréchet derivatives  $g_y^m$  and  $h_y^m$  respectively.

We locate *steady* symmetry breaking bifurcation points by constructing a discretized version of the extended system (25) as follows.

Find  $\mathbf{y} \in \mathbf{H}^{O(2)}$ ,  $\vec{\phi} \in \mathbf{V}_m^s$  and  $\lambda \in \mathbb{R}$  such that

$$\mathbf{G}(\mathbf{z}) = \begin{pmatrix} \mathbf{f} \\ \mathbf{g}_y^m \vec{\phi} \\ \mathbf{l}^T \vec{\phi} - 1 \end{pmatrix} = \mathbf{0}, \quad (42)$$

where  $\mathbf{z} = (\mathbf{y}, \vec{\phi}, \lambda) = (\mathbf{H}^{O(2)} \times \mathbf{V}_m^s \times \mathbb{R}) \in \mathbf{Z}$  and  $\mathbf{G} : \mathbf{Z} \mapsto \mathbf{Z}$ . Note that  $\dim(\mathbf{Z}) = 2N + 1$ .

We could equally well have chosen to solve the problem: find  $\mathbf{y} \in \mathbf{H}^{O(2)}$ ,  $\vec{\phi} \in \mathbf{V}_m^a$  and  $\lambda \in \mathbb{R}$  such that

$$\begin{pmatrix} \mathbf{f} \\ \mathbf{h}_y^m \vec{\phi} \\ \mathbf{l}^T \vec{\phi} - 1 \end{pmatrix} = \mathbf{0}.$$

The extended system (42) is a system of size  $(2N + 1)$  and it might seem that when solving (42) by Newton's method one would have to actually solve a sequence of  $(2N + 1) \times (2N + 1)$  linear systems. In fact, as was shown in Werner & Spence [13], these systems can be solved efficiently by a procedure involving the solution of a system of size  $N$  with two right hand sides, and a system of size  $(N + 1)$  with a single right hand side.

In many cases one wishes to calculate a *path* of symmetry-breaking bifurcations. We now describe an appropriate solution procedure to compute such a path based on the technique in Werner & Spence [13], which only

requires additional back-substitutions. Consider the two parameter problem  $f(y, \lambda, \sigma) = 0$ . Assume that for fixed  $\sigma$  a bifurcation point has been calculated using (42). Now as  $\sigma$  varies a path of symmetry-breaking bifurcations will be described, given by the solutions of the following system, where  $\mathbf{y} \in \mathbf{H}^{O(2)}$ ,  $\vec{\phi} \in \mathbf{V}_m^s$  and  $\lambda \in \mathbf{R}$ :

$$\mathbf{G}(\mathbf{z}, \sigma) = \begin{pmatrix} \mathbf{f}(\mathbf{y}, \lambda, \sigma) \\ \mathbf{g}_{\mathbf{y}}^m(\mathbf{y}, \lambda, \sigma)\vec{\phi} \\ \mathbf{l}^T \vec{\phi} - 1 \end{pmatrix} = \mathbf{0}, \quad (43)$$

where  $\mathbf{z} = (\mathbf{y}, \vec{\phi}, \lambda) = (\mathbf{H}^{O(2)} \times \mathbf{V}_m^s \times \mathbf{R}) \in \mathbf{Z}$  and  $\mathbf{G} : \mathbf{Z} \times \mathbf{R} \mapsto \mathbf{Z}$ .

The path following could be accomplished as follows. For an initial guess  $(\mathbf{z}_0, \sigma_0)$  such that  $\mathbf{G}(\mathbf{z}_0, \sigma_0) \neq \mathbf{0}$ , we seek  $\Delta \mathbf{z}$  and  $\Delta \sigma$  such that  $\mathbf{G}(\mathbf{z}_0 + \Delta \mathbf{z}, \sigma_0 + \Delta \sigma) = \mathbf{0}$ . Linearizing about  $(\mathbf{z}_0, \sigma_0)$ ,

$$\begin{aligned} \mathbf{G}(\mathbf{z}_0 + \Delta \mathbf{z}, \sigma_0 + \Delta \sigma) &\approx \mathbf{G}(\mathbf{z}_0, \sigma_0) + \mathbf{G}_{\mathbf{z}}(\mathbf{z}_0, \sigma_0)\Delta \mathbf{z} + \mathbf{G}_{\sigma}(\mathbf{z}_0, \sigma_0)\Delta \sigma \\ &= \mathbf{0} \end{aligned}$$

or

$$\mathbf{G}_{\mathbf{z}}(\mathbf{z}_0, \sigma_0)\Delta \mathbf{z} + \mathbf{G}_{\sigma}(\mathbf{z}_0, \sigma_0)\Delta \sigma = -\mathbf{G}(\mathbf{z}_0, \sigma_0),$$

Applied to (43) this is

$$\begin{pmatrix} \mathbf{f}_{\mathbf{y}} & 0 & \mathbf{f}_{\lambda} \\ \mathbf{g}_{\mathbf{y}\mathbf{y}}^m \vec{\phi} & \mathbf{g}_{\mathbf{y}}^m & \mathbf{g}_{\mathbf{y}\lambda}^m \vec{\phi} \\ 0 & \mathbf{l}^T & 0 \end{pmatrix} \begin{pmatrix} \Delta \mathbf{y} \\ \Delta \vec{\phi} \\ \Delta \lambda \end{pmatrix} + \begin{pmatrix} \mathbf{f}_{\sigma} \\ \mathbf{g}_{\mathbf{y}\sigma}^m \vec{\phi} \\ 0 \end{pmatrix} \Delta \sigma = - \begin{pmatrix} \mathbf{f} \\ \mathbf{g}_{\mathbf{y}}^m \vec{\phi} \\ \mathbf{l}^T \vec{\phi} - 1 \end{pmatrix},$$

where the derivatives are evaluated at  $(\mathbf{z}_0, \sigma_0)$ . Note that  $\mathbf{f}_{\mathbf{y}} = \mathbf{g}_{\mathbf{y}}^0$ , the Jacobian restricted to the  $O(2)$ -symmetric space  $\mathbf{H}^{O(2)}$ .

At a symmetry-breaking bifurcation point  $m \neq 0$ , so the matrix  $\mathbf{f}_{\mathbf{y}}$  is invertible on  $\mathbf{H}^{O(2)}$  and  $\mathbf{g}_{\mathbf{y}}^m$  is singular on  $\mathbf{V}_m^s$ . Letting

$$\Delta \mathbf{y} = \mathbf{a} + \mathbf{b}\Delta \lambda + \mathbf{c}\Delta \sigma, \quad (44)$$

the equation

$$\mathbf{f}_{\mathbf{y}}\Delta \mathbf{y} + \mathbf{f}_{\lambda}\Delta \lambda + \mathbf{f}_{\sigma}\Delta \sigma = -\mathbf{f} \quad (45)$$

becomes

$$\begin{aligned} \mathbf{f}_{\mathbf{y}}(\mathbf{a} + \mathbf{b}\Delta \lambda + \mathbf{c}\Delta \sigma) + \mathbf{f}_{\lambda}\Delta \lambda + \mathbf{f}_{\sigma}\Delta \sigma &= -\mathbf{f} \\ \text{or } \mathbf{f}_{\mathbf{y}}\mathbf{a} + (\mathbf{f}_{\mathbf{y}}\mathbf{b} + \mathbf{f}_{\lambda})\Delta \lambda + (\mathbf{f}_{\mathbf{y}}\mathbf{c} + \mathbf{f}_{\sigma})\Delta \sigma &= -\mathbf{f}, \end{aligned}$$

and we can solve

$$\mathbf{f}_y \mathbf{a} = -\mathbf{f}, \quad (46)$$

$$\mathbf{f}_y \mathbf{b} = -\mathbf{f}_\lambda, \quad (47)$$

$$\mathbf{f}_y \mathbf{c} = -\mathbf{f}_\sigma. \quad (48)$$

for  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{H}^{O(2)}$ . Thus we factorize a single  $N \times N$  dimensional matrix and solve for 3 different right hand sides.

Since  $\mathbf{g}_y^m$  is not invertible on  $V_m$  the second and third rows in the extended system,

$$\mathbf{g}_{yy}^m \vec{\phi} \Delta \mathbf{y} + \mathbf{g}_y^m \Delta \vec{\phi} + \mathbf{g}_{y\lambda}^m \Delta \lambda + \mathbf{g}_{y\sigma}^m \Delta \sigma = -\mathbf{g}_y^m \vec{\phi} \quad (49)$$

$$\mathbf{l}^T \Delta \vec{\phi} = \mathbf{l}^T \vec{\phi} - 1 \quad (50)$$

are solved together. Letting

$$\Delta \vec{\phi} = \mathbf{d} + \mathbf{e} \Delta \sigma \quad (51)$$

$$\Delta \lambda = \alpha + \beta \Delta \sigma \quad (52)$$

we have

$$\begin{aligned} \Delta \mathbf{y} &= \mathbf{a} + \mathbf{b} \Delta \lambda + \mathbf{c} \Delta \sigma \\ &= \mathbf{a} + \mathbf{b}(\alpha + \beta \Delta \sigma) + \mathbf{c} \Delta \sigma \\ &= (\mathbf{a} + \alpha \mathbf{b}) + (\beta \mathbf{b} + \mathbf{c}) \Delta \sigma. \end{aligned} \quad (53)$$

Substituting (51), (52) and (53) into (49), we have

$$\begin{aligned} \mathbf{g}_{yy}^m \vec{\phi} [(\mathbf{a} + \alpha \mathbf{b}) + (\beta \mathbf{b} + \mathbf{c}) \Delta \sigma] + \mathbf{g}_y^m (\mathbf{d} + \mathbf{e} \Delta \sigma) + \mathbf{g}_{y\lambda}^m \vec{\phi} (\alpha + \beta \Delta \sigma) \\ + \mathbf{g}_{y\sigma}^m \vec{\phi} \Delta \sigma = -\mathbf{g}_y^m \vec{\phi}. \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{g}_{yy}^m \vec{\phi} \mathbf{a} + \alpha \mathbf{g}_{yy}^m \vec{\phi} \mathbf{b} + \mathbf{g}_y^m \mathbf{d} + \alpha \mathbf{g}_{y\lambda}^m \vec{\phi} \\ + [\beta \mathbf{g}_{yy}^m \vec{\phi} \mathbf{b} + \mathbf{g}_{yy}^m \vec{\phi} \mathbf{c} + \mathbf{g}_y^m \mathbf{e} + \beta \mathbf{g}_{y\lambda}^m \vec{\phi} + \mathbf{g}_{y\sigma}^m \vec{\phi}] \Delta \sigma = -\mathbf{g}_y^m \vec{\phi}, \end{aligned}$$

or

$$\mathbf{g}_y^m \mathbf{d} + \alpha (\mathbf{g}_{yy}^m \vec{\phi} \mathbf{b} + \mathbf{g}_{y\lambda}^m \vec{\phi}) = -\mathbf{g}_y^m \vec{\phi} - \mathbf{g}_{yy}^m \vec{\phi} \mathbf{a}, \quad (54)$$

$$\text{and } \mathbf{g}_y^m \mathbf{e} + \beta (\mathbf{g}_{yy}^m \vec{\phi} \mathbf{b} + \mathbf{g}_{y\lambda}^m \vec{\phi}) = -\mathbf{g}_{y\sigma}^m \vec{\phi} - \mathbf{g}_{yy}^m \vec{\phi} \mathbf{c}. \quad (55)$$

Substituting (51) into (50), we have

$$\mathbf{l}^T(\mathbf{d} + \mathbf{e}\Delta\sigma) = \mathbf{l}^T\vec{\phi} - 1$$

or

$$\mathbf{l}^T\mathbf{d} = \mathbf{l}^T\vec{\phi} - 1 \tag{56}$$

$$\text{and } \mathbf{l}^T\mathbf{e} = 0. \tag{57}$$

Given  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  as solutions to (46), (47) and (48), we solve (54) and (56) together for  $\mathbf{d}$  and  $\alpha$ , and (55) and (57) together for  $\mathbf{e}$  and  $\beta$ . Note that we solve a single  $(N + 1) \times (N + 1)$  dimensional system with two different right hand sides. We then compute  $\Delta\vec{\phi}$  and  $\Delta\lambda$  and finally  $\Delta\mathbf{y}$  by substituting into (51), (52) and (53) respectively.

To compute  $d/b_\lambda$  as required in Lemma (2.3.1), we first solve  $\mathbf{f}_\mathbf{y}\mathbf{z} = \mathbf{g}_\mathbf{y}\mathbf{y}\vec{\phi}\vec{\phi}$  for  $\mathbf{z} \in \mathbf{H}^{O(2)}$ , then calculate the ratio

$$\frac{\psi^T(3\mathbf{g}_\mathbf{y}\mathbf{y}\vec{\phi}\mathbf{z} + \mathbf{g}_\mathbf{y}\mathbf{y}\mathbf{y}\vec{\phi}\vec{\phi}\vec{\phi})}{\psi^T(\mathbf{g}_\mathbf{y}\mathbf{y}\vec{\phi}\mathbf{b} + \mathbf{g}_\mathbf{y}\lambda\vec{\phi})}$$

where  $\mathbf{f}_\mathbf{y}\mathbf{b} = -\mathbf{f}_\lambda$ , see (47). It is worth mentioning that a change in the normalization of  $\phi$  will change the value of  $d/b_\lambda$ , but does not change its sign, since if  $\phi \rightarrow \beta\phi$  for some real  $\beta$ , then  $\mathbf{z} \rightarrow \beta^2\mathbf{z}$  and  $d/b_\lambda \rightarrow \beta^2d/b_\lambda$ .

Starting values for the Newton iteration were determined from the previous linear stability calculations reported by Tavener [8].

## 5 Comparison of theory in §2 to standard approach

Kim & Pearlstein [33], Natarajan & Acrivos [6], and Tavener [8] carried out a stability analysis for the problem in §3. Let

$$(u_r, u_\theta, u_z, p) = (u_r^0(r, z), 0, u_z^0(r, z), p^0(r, z)) \in H^{O(2)}$$

denote a steady  $O(2)$ -symmetric solution of equations (29)–(32). These authors considered perturbations of the form

$$u_r = u_r^0(r, z) + \epsilon u_r^1(r, z)e^{im\theta - \mu t}, \dots \text{etc.},$$

and derived a *complex* matrix eigenvalue problem for each  $m$ . In all three papers it was assumed or stated without proof that the perturbations do not introduce any mixed mode terms in the linearization. Since the linearization satisfies the equivariance condition (10), Theorem 2.2.1 applies and shows that this assumption is indeed correct. In addition, for  $m \neq 0$ , Tavener [8] showed how the complex matrix eigenvalue problem may be transformed to a real eigenvalue problem by simple manipulation, namely by multiplying the “ $u_\theta^1$ ” equation by  $i$ . Lemma 2.2.1 (b) shows that it will always be possible to make a transformation to a real matrix eigenvalue problem for any  $O(2)$ -symmetric problem.

In fact we now give a stability analysis that produces the real matrix eigenvalue problem directly, using our knowledge of the decomposition of  $H$  induced by the action of  $O(2)$  given by (38).

Consider a nearby solution of (29)–(32) the form

$$\begin{aligned} (u_r^0, 0, u_z^0, p^0) &+ \epsilon [u_r^1(r, z)e^{-\mu t}(\cos m\theta + i \sin m\theta), \\ &u_\theta^1(r, z)e^{-\mu t}(\sin m\theta + i \cos m\theta), \\ &u_z^1(r, z)e^{-\mu t}(\cos m\theta + i \sin m\theta), \\ &p^1(r, z)e^{-\mu t}(\cos m\theta + i \sin m\theta)], \end{aligned} \quad (58)$$

where  $0 < \epsilon \ll 1$ ,  $m$  is a positive integer and  $\mu$  is complex. Substituting the new solution into the equilibrium equations (29)–(32) and equating to zero terms that are real and first-order in  $\epsilon$ , the linearized equations are

$$R \left( -\mu u_r^1 + u_r^0 \frac{\partial u_r^1}{\partial r} + u_r^1 \frac{\partial u_r^0}{\partial r} + u_z^0 \frac{\partial u_r^1}{\partial z} + u_z^1 \frac{\partial u_r^0}{\partial z} \right)$$

$$\begin{aligned}
& + \frac{\partial p^1}{\partial r} \\
& - \left( \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u_r^1}{\partial r}) - \frac{m^2}{r^2} u_r^1 + \frac{\partial^2 u_r^1}{\partial z^2} - \frac{u_r^1}{r^2} - \frac{2m}{r^2} u_\theta^1 \right) = 0, \tag{59}
\end{aligned}$$

$$\begin{aligned}
& R \left( -\mu u_\theta^1 + u_r^0 \frac{\partial u_\theta^1}{\partial r} + u_z^0 \frac{\partial u_\theta^1}{\partial z} + \frac{u_r^0 u_\theta^1}{r} \right) \\
& - \frac{m}{r} p^1 \\
& - \left( \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u_\theta^1}{\partial r}) - \frac{m^2}{r^2} u_\theta^1 + \frac{\partial^2 u_\theta^1}{\partial z^2} + \frac{2m}{r^2} u_r^1 - \frac{u_\theta^1}{r^2} \right) = 0, \tag{60}
\end{aligned}$$

$$\begin{aligned}
& R \left( -\mu u_z^1 + u_r^0 \frac{\partial u_z^1}{\partial r} + u_r^1 \frac{\partial u_z^0}{\partial r} + u_z^0 \frac{\partial u_z^1}{\partial z} + u_z^1 \frac{\partial u_z^0}{\partial z} \right) \\
& + \frac{\partial p^1}{\partial z} \\
& - \left( \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u_z^1}{\partial r}) - \frac{m^2}{r^2} u_z^1 + \frac{\partial^2 u_z^1}{\partial z^2} \right) = 0, \tag{61}
\end{aligned}$$

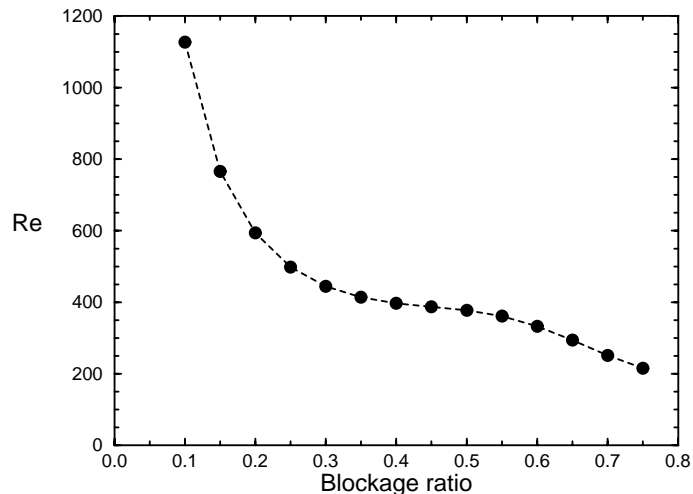
$$\frac{1}{r} \frac{\partial}{\partial r} (r u_r^1) + \frac{m}{r} u_\theta^1 + \frac{\partial u_z^1}{\partial z} = 0. \tag{62}$$

which involve no imaginary terms. An *identical* set of equations is obtained by equating the terms that are imaginary and first order in  $\epsilon$  to be zero.

## 6 Results

The critical Reynolds number  $Re$  obtained at a regular solution of extended system (42) for the flow past a centrally located sphere in a pipe is given as a function of blockage ratio in Figure 2.

Figure 2



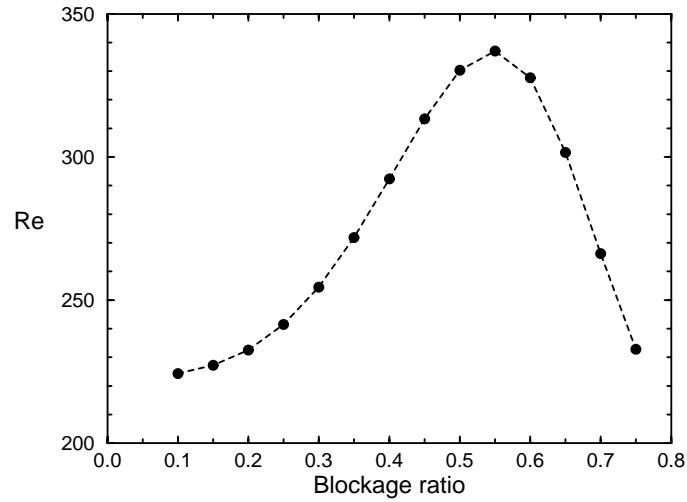
*Critical Reynolds number,  $Re$  vs. blockage ratio for the flow past a sphere in a pipe.*

For the purposes of comparison with experiments and computations of the exterior flow past spheres in “unbounded” domains, we define a second (dependent) Reynolds number  $Re_d$  based on the sphere diameter, and the axial inlet velocity  $u_z^*(r)$  averaged across the sphere cross-section. Let  $Re_d = d\bar{U}_d/\nu$ , where

$$\bar{U}_d = \frac{4}{\pi d^2} \int_0^{d/2} 2\pi r u_z^*(r) dr ,$$

with  $u_z^*(r)$  as in section 3. The corresponding critical Reynolds number  $Re_d$  are given as a function of blockage ratio in Table I and Figure 3.

Figure 3



*Critical Reynolds number,  $Re_d$  vs. blockage ratio  
for the flow past a sphere in a pipe*

Our results are consistent with those of Tavener [8], Figure 5. A finite element mesh with length 3 pipe diameters upstream of the sphere and 8 pipe diameters downstream of the sphere incorporating 4400 quadrilateral elements was used to obtain these values. Based on the studies shown in Tables II and III, we believe the critical Reynolds number to be accurate to within 1%.

Br	Critical $Re_d$	Br	Critical $Re_d$
0.10	224.4	0.45	313.4
0.15	227.2	0.50	330.3
0.20	232.6	0.55	337.0
0.25	241.5	0.60	327.7
0.30	254.6	0.65	301.6
0.35	271.9	0.70	266.3
0.40	292.4	0.75	232.8

Table I  
*Critical  $Re_d$  vs blockage ratio.*

The result of two consecutive mesh halvings is given in Table II for a blockage ratio of 0.5.

No. of elements	Critical $Re_d$	$d/b_\lambda$
275	343.4	-34.83
1100	333.0	-31.87
4400	330.3	-33.39

Table II  
*Convergence study for blockage ratio 0.5.*

It is evident from Table III, that the effect of domain length downstream of the sphere is weak and that even for very short domains, the estimate of the critical Reynolds number is surprisingly good. This, and the null eigenvectors themselves (as presented previously by both Natarajan and Acrivos [6] and Tavener [8]) indicate that the instability is localized in the sphere wake.

Domain length downstream of sphere (pipe diameters)	Critical $Re_d$
3.5	233.6
5.5	233.7
8	233.8
10.5	234.0
13	234.2

Table III  
*Effect of domain length for blockage ratio 0.75  
using a mesh with 1331 elements.*

As predicted by the theory in section 2, the determinant of the Jacobian  $\mathbf{f}_y$  did *not* change sign along the  $O(2)$ -symmetric solution branches at the symmetry-breaking bifurcation points. Previous eigenvalue computations by [8] were necessary to provide an initial guess for the extended system (42). An assumption of smoothness with respect to the blockage ratio was used to provide an initial guess for the critical Reynolds number for blockage ratios at which eigenvalue computation had not been performed.

The sign of the ratio  $d/b_\lambda$  (Table II and indeed for all computations) indicates that these pitchfork bifurcations are super-critical (see Lemma 2.3.1), a result that is consistent with both the experimental evidence of Margarvey & Bishop [3], Nakamura [4] and Wu & Faeth [5], the computational evidence of Tomboulides, Orszag & Karniadakis, [7] and the conjecture of Natarajan and Acrivos [6]. It could not be determined by the earlier eigenvalue computations alone.

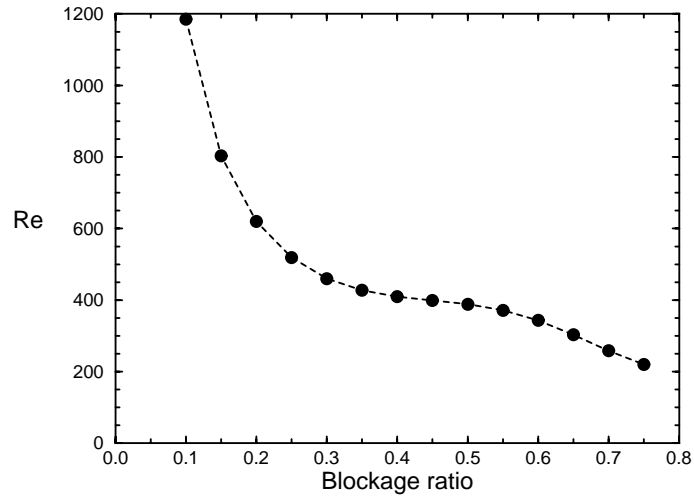
Wham et al. [11] show that the presence of a rear-facing sting or wire that is sometimes used in experiments to support a sphere, makes only a 3% difference to the drag on the sphere. Encouraged by this result, we repeated our computational procedure setting  $u = v = 0$  along the centerline of the pipe downstream of the sphere to model the presence of a wire. As can be seen from Table IV, we observe change in the critical Reynolds number at the  $O(2)$ -symmetry breaking bifurcation point of between 3% and 5%. The bifurcation remains supercritical in nature for all blockage ratios examined.

Br	Critical $Re_d$	Br	Critical $Re_d$
0.10	235.9	0.45	322.7
0.15	238.2	0.50	339.9
0.20	243.0	0.55	346.8
0.25	251.3	0.60	337.5
0.30	263.9	0.65	310.4
0.35	281.0	0.70	273.1
0.40	301.5	0.75	237.6

Table IV  
*Critical  $Re_d$  vs blockage ratio.*

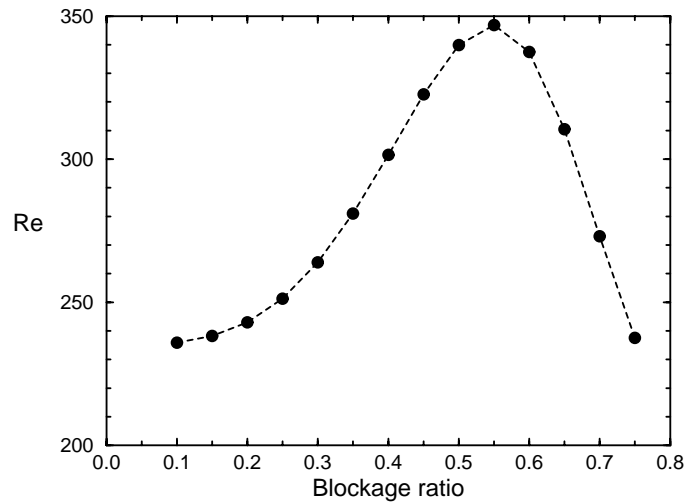
Plots of the critical Reynolds numbers  $Re$  and  $Re_d$  as a function of blockage ratio are shown in Figures 4 and 5 respectively. These results were obtained on the same grid as that used for the unsupported sphere.

Figure 4



*Critical Reynolds number,  $Re$  vs. blockage ratio for the flow past a sphere with a downstream sting.*

Figure 5



*Critical Reynolds number,  $Re_d$  vs. blockage ratio for the flow past a sphere with a downstream sting.*

Based on a second convergence study (Table V) we again believe that the critical Reynolds numbers are accurate to within 1%.

No. of elements	Critical $Re_d$	$d/b_\lambda$
275	356.7	-71.63
1100	344.6	-43.38
4400	339.9	-43.51

Table V  
*Convergence study for blockage ratio 0.5.*

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