Introduction to Voting Theory

Arrow's Impossibility Theorem

A method for determining election results that is democratic and always fair is a mathematical impossibility.

Ballots- how individual voters express opinions

⁕ Top Choice Ballot- most familar, voter picks first choice only

⁕ Preference Ballot- not the most common type, choices listed in order of preference

Example:

List the following fruits in order of preference: Apple, Banana, Mango, Pear: ❶ Mango ❷ Banana ❸ Apple ❹ Pear

Two important voting properties to use with Preference Ballots:

Transitivity of individual preferences

if a voter prefers **A** to **B** and **B** to **C**, then the voter prefers **A** to **C**.

Elimination of a Candidate

If a voter ranks candidates **A, B, C, D** and candidate **B** drops out of the election, then the new rank is **A, C, D**. (i.e. relative preferences are preserved)

Using preference ballots

Some sample ballots:

	Ballot		Ballot		Ballot		Ballot		Ballot
1^{st}	M	1^{st}	B	1^{st}	M	1^{st}	M	1^{st}	B
2^{nd}	P	2^{nd}	M	2^{nd}	P	2^{nd}	A	2^{nd}	M
3^{rd}	A	3^{rd}	A	3^{rd}	A	3^{rd}	P	3^{rd}	A
4^{th}	B	4^{th}	P	4^{th}	B	4^{th}	B	4^{th}	P

Make a **Preference Schedule**

Step 1: combine identical ballots

Ballot		Ballot		Ballot		Ballot		Ballot	
1^{st}	M	1^{st}	M	1^{st}	B	1^{st}	B	1^{st}	M
2^{nd}	P	2^{nd}	P	2^{nd}	M	2^{nd}	M	2^{nd}	A
3^{rd}	A	3^{rd}	A	3^{rd}	A	3^{rd}	A	3^{rd}	P
4^{th}	B	4^{th}	B	4^{th}	P	4^{th}	P	4^{th}	B

Step 2: Organize results in a table

Preference Schedule: Favorite Fruit

Number of voters	**2**	**2**	**1**
First choice	**M**	**B**	**M**
Second choice	**P**	**M**	**A**
Third choice	**A**	**A**	**P**
Fourth choice	**B**	**P**	**B**

Consider another election: The Math Appreciation Society is voting for president. The **candidates** are **A**lisha, **B**oris, **C**armen, and **D**ave. 37 club members vote, using a preference ballot.

Summary of the 37 ballots:

Preference Schedule: MAS Election

Number of voters	14	10	8	4	1
First choice	A	C	D	B	C
Second choice	B	B	C	D	D
Third choice	C	D	B	C	B
Fourth choice	D	A	A	A	A

Plurality Method

Candidate with the most first place votes wins.

Plurality vs. Majority

Majority- more than half of the votes

Plurality- the most first place votes

The Majority Criterion

If a choice receives a majority of the first-place votes in an election, then that choice should be the winner of the election.

Plurality method satisfies the majority criterion-

The marching band is deciding which bowl to play at (Rose, Fiesta, Hula, Orange, Sugar). Here is the preference schedule summarizing the ballots.

Preference Schedule: Which Bowl?

Number of voters	49	48	3
First choice	R	H	F
Second choice	H	S	H
Third choice	F	O	S
Fourth choice	O	F	O
Fifth choice	S	R	R

Condorcet Criterion

If there is a choice that in a head-to-head comparison is preferred by the voters over every other choice, then that choice should be the winner of the election.

Head-to-head comparison: Compare two candidates, then another two, until all candidates have been considered. Is there one candidate that is always preferred?

Preference Schedule: Which Bowl?

Number of voters	**49**	**48**	**3**
First choice	**R**	**H**	**F**
Second choice	**H**	**S**	**H**
Third choice	**F**	**O**	**S**
Fourth choice	**O**	**F**	**O**
Fifth choice	**S**	**R**	**R**

Call the Hula Bowl a **Compromise Candidate**

Insincere Voting- problem with plurality voting

Borda Count

- ✦ looks at all positions, not just first place
- ✦ compromise candidate
- ✦ preference schedule

The Borda Count works by assigning points for places. Four places:

first place gets 4 points,

second place gets 3 points,

third place gets 2 points and

fourth place gets 1 point.

Add up all the points for each candidate and the winner is the candidate with the most points.

Example: Favorite Fruit

Ballot		Ballot		Ballot		Ballot		Ballot	
1^{st}	M	1^{st}	B	1^{st}	M	1^{st}	M	1^{st}	B
2^{nd}	P	2^{nd}	M	2^{nd}	P	2^{nd}	A	2^{nd}	M
3^{rd}	A	3^{rd}	A	3^{rd}	A	3^{rd}	P	3^{rd}	A
4^{th}	B	4^{th}	P	4^{th}	B	4^{th}	B	4^{th}	P

Let's add points for each fruit: Remember, 4 points for each first place vote, 3 for each second place, etc.

Mango: $4 + 3 + 4 + 4 + 3 = 18$ points

Banana: $1 + 4 + 1 + 1 + 4 = 11$ points

Apple: $2 + 2 + 2 + 3 + 2 = 11$ points

Pear: $3 + 1 + 3 + 2 + 1 = 10$ points

Winner is Mango.

How do we do the Borda Count if we only have a preference schedule?

Use (#voters) $\times$ (points for the position) for each column and then add.

Use the Borda Count Method to determine the winner of the MAS Election.

Preference Schedule: MAS Election

Number of voters	**14**	**10**	**8**	**4**	**1**
First choice	**A**	**C**	**D**	**B**	**C**
Second choice	**B**	**B**	**C**	**D**	**D**
Third choice	**C**	**D**	**B**	**C**	**B**
Fourth choice	**D**	**A**	**A**	**A**	**A**

A: $\mathbf{14 \times 4 + 10 \times 1 + 8 \times 1 + 4 \times 1 + 1 \times 1 = 79}$

B: $\mathbf{14 \times 3 + 10 \times 3 + 8 \times 2 + 4 \times 4 + 1 \times 2 = 106}$

C: $\mathbf{14 \times 2 + 10 \times 4 + 8 \times 3 + 4 \times 2 + 1 \times 4 = 104}$

D: $\mathbf{14 \times 1 + 10 \times 2 + 8 \times 4 + 4 \times 3 + 1 \times 3 = 81}$

Boris is winner!

School Principal Example

A school needs to elect a new principal.

Candidates: Mrs. Amaro, Mr. Burr, Mr. Castro, and Ms. Dunbar

Preference Schedule: Principal

Number of voters	6	2	3
First choice	A	B	C
Second choice	B	C	D
Third choice	C	D	B
Fourth choice	D	A	A

Try it: Use the Borda Count to find the winner.

B, or Mr. Burr is winner.

<u>**Summary**</u>

✦Two Ballot Types, Top Choice and Preference

✦Preference Schedule summarizes the ballots

✦Arrow's Impossibility Theorem: It is impossible to fairly and democratically pick a winner.

✦Plurality Method for chosing winner picks the candidate with the most first place votes.

✦The Plurality Method satisfies the Majority Criterion.

✦The Plurality Method can violate the Condorcet Criterion.

✦Insincere Voting

✧ Borda Count- In an election with N candidates we give 1 point for last place, 2 points for second from last place,..., and N points for first place. The choice with the highest total wins.

✧ Can violate the Majority Criterion

✧ Can violate the Condorcet Criterion

✧ Finds the best compromise candidate.

✧ Used for the Heisman Award, American and National Baseball MVP, Country Music Vocalist of the Year