In the voting systems we have used so far in this class, each person has one vote.

We can call this a <u>One Person, One Vote</u> system.

Now consider examples where one person controls more than one vote. Call this:

<u>Weighted Voting:</u> One person - x votes

Example: Electoral College

- **Each state gets a number of votes equal to the number of state representatives and senators.**
- **Colorado has 7 representatives and 2 senators for 9 electoral votes**
- **California has 55 electoral votes**
- **Alaska, Montana and District of Columbia each get 3 electoral votes**

The votes are not split- for example, all 55 of California's votes go for the candidate who won the popular vote.
(Colorado's last election- amendment to change this.)
Certain states will have more power than other states, for example, California vs. Colorado.
Became very noticable in the 2000 and 2004 Presidential Elections (Florida and Ohio)
One state had the power to determine the outcome of the election.

Use Mathematical Ideas to measure the *power* in a weighted voting system

<u>Other examples of weighted systems:</u>

- **division of power in the United Nations Security Council**
- **county board of supervisors**
- **corporations**

Weighted Voting System- formal voting arrangement in which the voters are not necessarily equal in terms of the number of votes they control

Consider elections with two choices only

<u>Motion</u>- vote involving 2 choices, or a *yes-no* vote

Three elements: *players, weights, quota*

<u>Players</u>- the voters (denoted $P_1, P_2, ..., P_N$)

<u>Weights</u>- how many votes each player controls ($w_1, w_2, ..., w_N$)

<u>Quota</u>- minimum number of votes needed to pass a motion (q)

How big is the quota? This can vary and can be any number greater than half the total number of votes but not more than the total number of votes.

How Big is the Quota? (Math answer)

$$\frac{w_1 + w_2 + \cdots + w_N}{2} < q \leq w_1 + w_2 + \cdots + w_N$$

In other words, the smallest possible quota is the majority.
The largest possible quota is all the possible votes added together.

Notation: $[q : w_1, w_2, ..., w_N]$

Example: Consider a corporation with four share holders: P_1, P_2, P_3, P_4
P_1 **has 8 votes,** P_2 **has 6 votes,** P_3 **has 5 votes and** P_4 **has 1 vote. The Corporation's bylaws state that two-thirds of the 20 votes are needed to pass a motion.**

Total of 8 + 6 + 5 + 1 = 20 votes.
Two-thirds is 13.33. Quota is 14.

Description: $[14 : 8, 6, 5, 1]$

Example: Consider a corporation with four share holders. P_1 **has 5 votes,** P_2 **has 4 votes,** P_3 **has 4 votes and** P_4 **has 2 votes. Suppose the quota is 7.**

Description: $[7 : 5, 4, 4, 2]$

Question: What happens if P_1 and P_4 vote yes and P_2 and P_3 vote no? Note: Quota is too small. (15 total votes, quota needs to be at least 8)

Example: $[17 : 5, 4, 4, 2]$

Question: What is wrong with this example?

Quota is too big- no motion could be passed.

Example: Compare $[11 : 4, 4, 4, 4, 4]$ **and** $[3 : 1, 1, 1, 1, 1]$

They are the same. Note that both need at least 3 players to pass a motion.

Example: $[15 : 5, 4, 3, 2, 1]$

Question: How many votes needed to pass a motion?

All votes are needed- Unanimous.

Same as $[5 : 1, 1, 1, 1, 1]$

one person- one vote in disguise.

Example: $[11 : 12, 5, 4]$

Note that P_1 has all the power!

Dictator- player's weight greater than or equal to quota

Dummy- player with no power

Note that P_1 is the Dictator, and P_2 and P_3 are "dummy" players

Example: $[12 : 9, 5, 4, 2]$

Note that P_1 is not a dictator, but does have the power to stop any motion from passing. Even if P_2, P_3, P_4 voted together, they wouldn't have enough votes to pass the motion.

<u>Veto Power</u>- player that has power to prevent passage of a motion

Example 2.8 $[101 : 99, 98, 3]$

Note that all players have equal power.

Which sets of players can combine votes and carry a motion?

- P_1 **and** P_2 **control 197 votes**
- P_1 **and** P_3 **control 102 votes**
- P_2 **and** P_3 **control 101 votes**
- P_1, P_2, **and** P_3 **control all votes**

Note that no single player has enough power to carry a motion.

<u>coalition</u>- set of players that might vote together

Notation: $\{P_1, P_2\}$ **is the set (or coalition) containing** P_1 **and** P_2

$\{P_2\}$ **set containing** P_2

weight of the coalition- total number of votes controlled by a coalition

winning coalitions- have enough votes to win

losing coalitions- don't have enough votes to win

grand coalition- coalition of all players, always wins

empty set- coalition of no players, denoted $\{\emptyset\}$

critical player- a person whose decision could turn a winning coalition into a losing coalition

Summary of Coalitions for $[101 : 99, 98, 3]$

	Coalition	Weight	W or L	Critical Players
1	$\{\emptyset\}$	0	N/A	N/A
2	$\{P_1\}$	99	Lose	N/A
3	$\{P_2\}$	98	Lose	N/A
4	$\{P_3\}$	3	Lose	N/A
5	$\{P_1, P_2\}$	197	Win	P_1 and P_2
6	$\{P_1, P_3\}$	102	Win	P_1 and P_3
7	$\{P_2, P_3\}$	101	Win	P_2 and P_3
8	$\{P_1, P_2, P_3\}$	200	Win	None

<u>Banzhaf Power Index</u>- measures a player's power with the idea that the more the player is critical, the more power that player holds

In the last example, since all three players are critical the same number of times, they each hold one-third of the power.
Each has a Banzhaf power index equal to $\frac{1}{3}$.

Finding the Banzhaf Power Index of Player P

- **Step 1. Make a list of all possible coalitions**
- **Step 2. Determine the winning coalitions.**
- **Step 3. In each winning coalition, determine which of the players are *critical* players.**
- **Step 4. Count the total number of times a player P is critical.**
- **Step 5. Count the total number of times all players are critical.**

The Banzhaf power index of player P is then given by the fraction:

$$\frac{\textbf{number of times a player is critical}}{\textbf{number of times all players are critical}}$$

Example:

$[4:3,2,1]$ **Find the Banzhaf power index of each player.**

- **Step 1.**

  8 coalitions: $\{\emptyset\}$, $\{P_1\}$, $\{P_2\}$,$\{P_3\}$, $\{P_1, P_2\}$, $\{P_1, P_3\}$, $\{P_2, P_3\}$, $\{P_1, P_2, P_3\}$

- **Step 2.**

Winning Coalitions	Critical Players
$\{P_1, P_2\}$	P_1 **and** P_2
$\{P_1, P_3\}$	P_1 **and** P_3
$\{P_1, P_2, P_3\}$	P_1

- **Step 3.** P_1 **is critical 3 times.** P_2 **is critical 1 times.** P_3 **is critical 1 times.**

- **Step 4.** $3+1+1=5$ **so there are 5 times when the players are critical.**

The Banzhaf power index of each player is: $P_1 : \frac{3}{5}$, $P_2 : \frac{1}{5}$, $P_3 : \frac{1}{5}$

P_2 **and** P_3 **have the same amount of power.**

Banzhaf power distribution- listing of Banzhaf power indexes.

$P_1 : 60\%$, $P_2 : 20\%$, $P_3 : 20\%$

Basketball team, the Akron Flyers, are drafting players. The coach, manager, scout, and trainer will use a weighted voting system: $[6:4,3,2,1]$

We can list coalitions with weights instead of player names. (worksheets)

Coalition	Weight	Win or Lose
$\{\emptyset\}$	N/A	N/A
$\{4\}$	4	Lose
$\{3\}$	3	Lose
$\{2\}$	2	Lose
$\{1\}$	1	Lose
$\{\underline{4},\underline{3}\}$	7	Win
$\{\underline{4},\underline{2}\}$	6	Win
$\{4,1\}$	5	Lose
$\{3,2\}$	5	Lose
$\{3,1\}$	4	Lose
$\{2,1\}$	3	Lose
$\{\underline{4},3,2\}$	9	Win
$\{\underline{4},\underline{3},1\}$	8	Win
$\{\underline{4},\underline{2},1\}$	7	Win
$\{\underline{3},\underline{2},\underline{1}\}$	6	Win
$\{4,3,2,1\}$	10	Win

Banzhaf Power Distribution:

Coach:$\frac{5}{12}=41\frac{2}{3}\%$**, Manager:**$\frac{3}{12}=\frac{1}{4}=25\%$**,**
Scout:$\frac{3}{12}=\frac{1}{4}=25\%$**, Trainer:**$\frac{1}{12}=8\frac{1}{3}\%$

Notice that the fractions can be reduced.

How Many Coalitions? **(note: book different)**

Consider two players: $\{P_1, P_2\}$

4 subsets: $\{\}$, $\{P_1\}$, $\{P_2\}$, $\{P_1, P_2\}$.

Consider three players: $\{P_1, P_2, P_3\}$

We have the 4 from the previous set above, and also the same subsets but with P_3 **added.**

$\{\}$, $\{P_1\}$, $\{P_2\}$, $\{P_1, P_2\}$ ***and***

$\{P_3\}$, $\{P_1, P_3\}$,$\{P_2, P_3\}$,$\{P_1, P_2, P_3\}$.

Four players: all above subsets *and* all sets with P_4 **added.**

Notice that we are doubling the number of subsets each time.

Players	Subsets
P_1	2
P_1, P_2	$2 \times 2 = 2^2$
P_1, P_2, P_3	$2 \times 4 = 2^3$
P_1, P_2, P_3, P_4	$2 \times 8 = 2^4$
$\vdots$	$\vdots$
$P_1, P_2, ..., P_N$	2^N

If N **is the number of players, then there are** 2^N **coalitions, including the empty set.**

Nassau County Board of Supervisors, New York

Nassau County has 6 districts with 115 votes total and a quota of 58 votes.

District	votes in 1964
Hempstead #1	31
Hempstead #2	31
Oyster Bay	28
North Hempstead	21
Long Beach	2
Glen Cove	2

$[58 : 31, 31, 28, 21, 2, 2]$

- ✧ **No winning coalition possible without one of the first three members.**
- ✧ **North Hempstead, Long Beach, and Glen Cove without power**
- ✧ **John Banzhaf introduced the Banzhaf power index in 1965 to identify these problems with the Nassau County Board of Supervisors.**

<u>**United Nations Security Council**</u>

- **Has 15 members**
- **5 are permanent (Britain, China, France, Russia and the US) and have equal power**
- **10 are nonpermanent (rotate on a 2 year term) and have equal power**

For a motion to pass: needs a yes from each permanent member and yes votes from at least 4 of the nonpermanent members

- **848 winning coalitions**
- **210 winning coalitions where nonpermanent members are critical**
- **5080 total times all players are critical**

Banzhaf power index for permanent member:

848/5080 **or about 16.7%**

Banzhaf power index for nonpermanent member:

210/5080 **or about 1.65%**