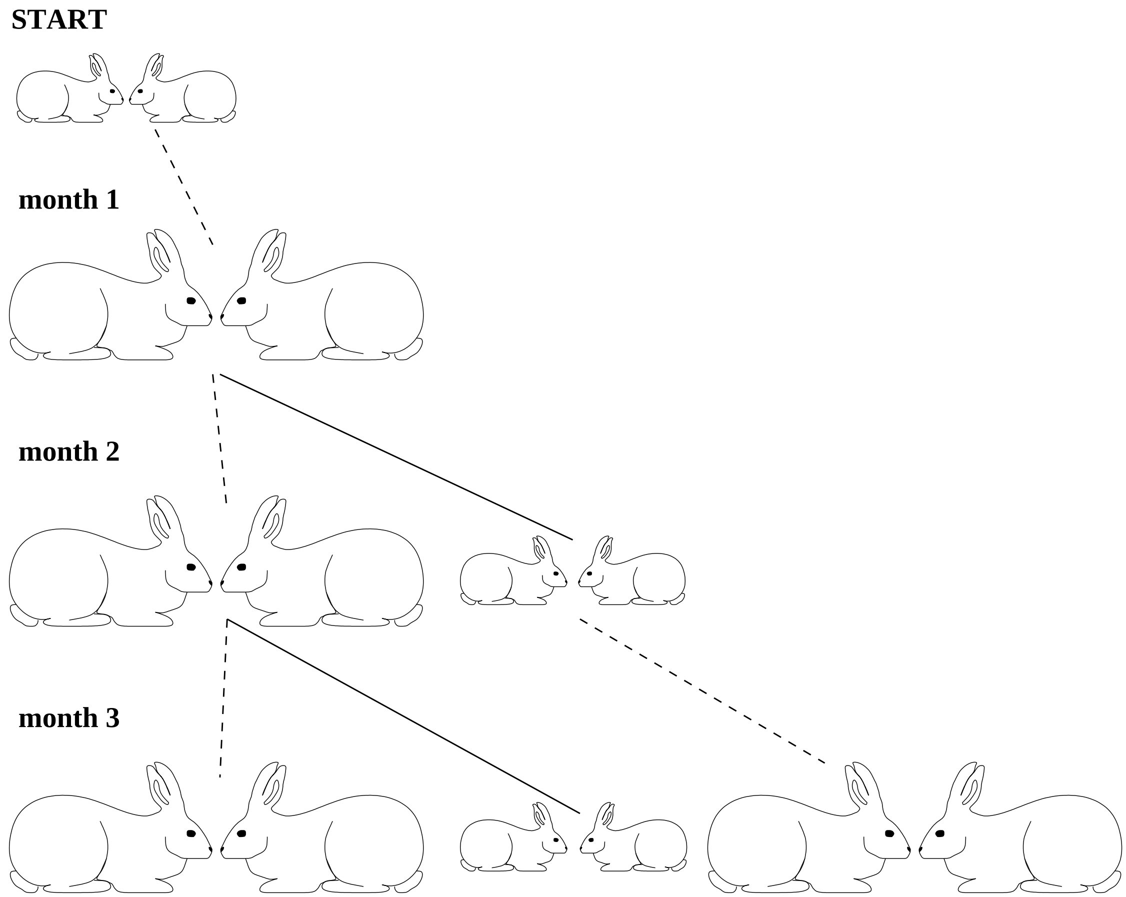
Chapter 9- Fibonacci Numbers

Example: Rabbit Growth

- Start with 1 pair of rabbits
- One month to maturity
- Mature pairs produce 1 pair of offspring each month

How Many Pairs of Rabbits?

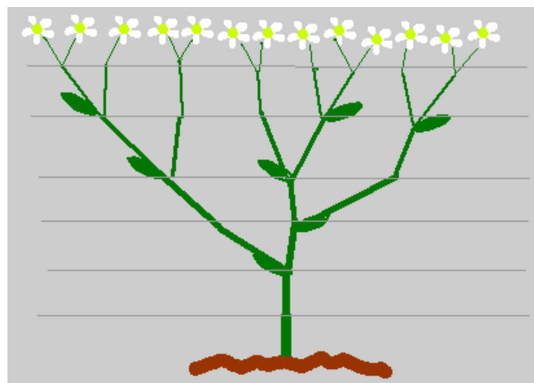


1,1,2,3,5,8,13,....

Is this a realistic growth model?

Count things in nature:

Flower petals



Numbers that we found:

Petals, bracts and leaves:

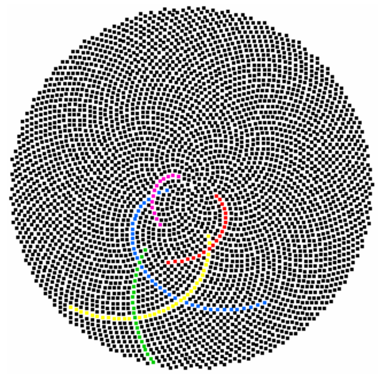
8, 13, 13, 3, 5, 5,

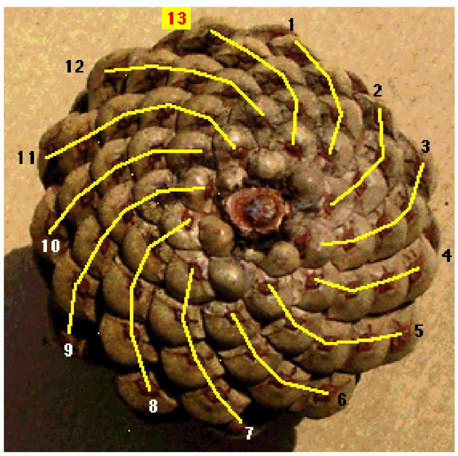
Branches:

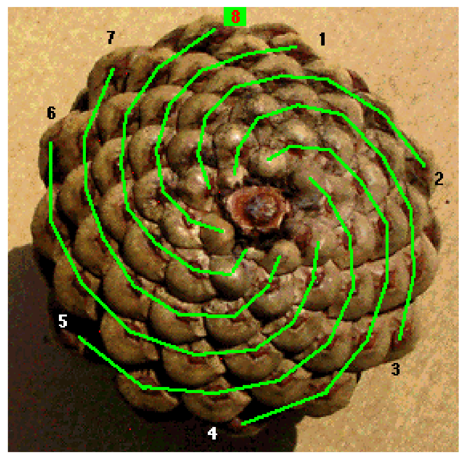
1, 1, 2, 3, 5, 8, 13

Some other flowers have 55 or 89 petals.

Count spirals in flowers and pinecones







Numbers found (pinecones): 8 and 13

List all the numbers in order:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89

These numbers are also found in the Fibonacci

Sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Is there a pattern?

Describe the pattern using a **Recursive Rule**

recursive rule- defines a number using eariler numbers in the sequence

Define each term as F_N

Example: $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, and $F_4 = 3$

A recursive definition for the Fibonacci Numbers:

Let $F_1 = 1$ and $F_2 = 1$ be the starting values (seeds) for the sequence.

$F_{N+1} = F_{N-1} + F_N$ for $N \geq 3$

Try it with the Fibonacci Sequence:

$F_1 = 1$, $F_2 = 1$ so $F_3 = F_1 + F_2 = 1 + 1 = 2$

$F_4 = F_2 + F_3 = 1 + 2 = 3$

Question: If $F_{13} = 233$ and $F_{14} = 377$, what is F_{15}?

$F_{15} = 233 + 377 = 610$

Back to the Fibonacci Sequence. How could we find F_{42}?

The only way we know as of now is to step our way there recursively, but we'd need to find all 41 previous Fibonacci Numbers.

Is there a short cut?

A recursive definition depends on all previous values. An explicit definition allows calculation of a specific number without needing all previous values.

An explicit definition for Fibonacci Numbers:

Binet's Formula

$$F_N = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^N - \left(\frac{1-\sqrt{5}}{2}\right)^N}{\sqrt{5}}$$

Examples:

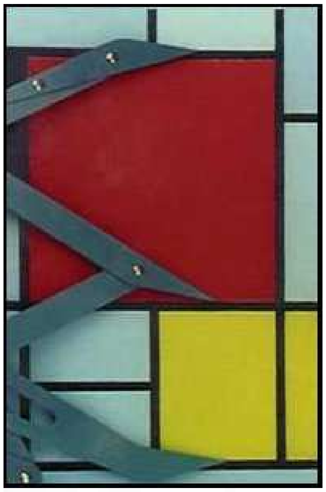
$$F_{42} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{42} - \left(\frac{1-\sqrt{5}}{2}\right)^{42}}{\sqrt{5}}$$

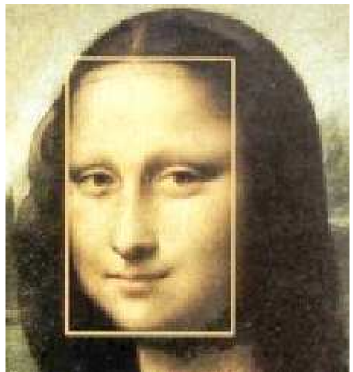
$F_{42} = 267,914,296$

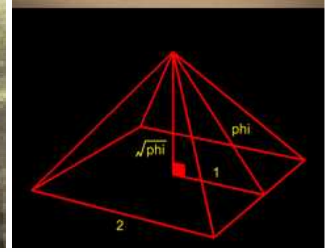
F_{15}:

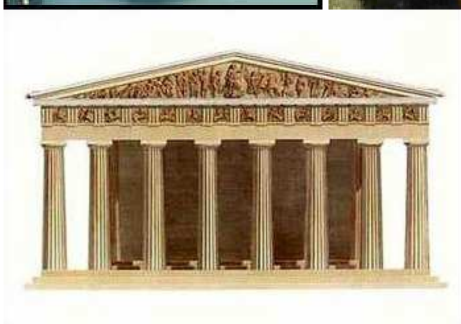
$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{15} - \left(\frac{1-\sqrt{5}}{2}\right)^{15}}{\sqrt{5}} = 610$$

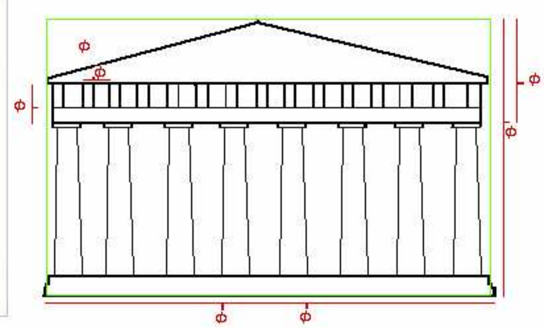
Look at ratios in Art and Architecture





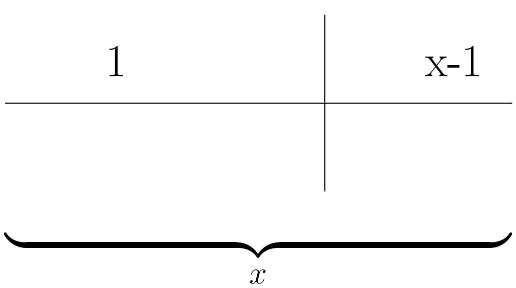






Find this ratio:

Consider the following diagram and the ratio of the lengths:



$$\frac{1}{x-1} = \frac{x}{1}$$

This leads us to the following formula: $x^2 = x + 1$

We can us the quadratic formula to solve this equation for x:

1) Rewrite $x^2 - x - 1 = 0$
2) Use $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$
\begin{aligned}
x &= \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \\
x &= \frac{1 \pm \sqrt{1 + 4}}{2} \\
x &= \frac{1 \pm \sqrt{5}}{2}
\end{aligned}
$$

$\dfrac{1 + \sqrt{5}}{2} \approx 1.618$ and $\dfrac{1 - \sqrt{5}}{2} \approx -0.618$

Ratio: 1.618

Look at ratios of consecutive Fibonacci Numbers:

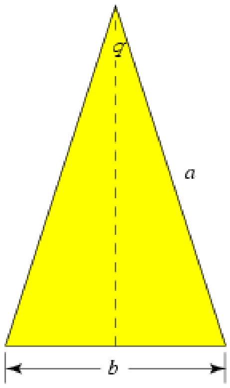
$\frac{55}{34} \approx 1.61764$ $\qquad$ $\frac{89}{55} \approx 1.61818$ $\qquad$ $\frac{144}{89} \approx 1.61798$

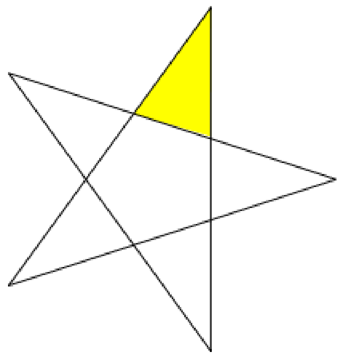
As the Fibonacci numbers get larger, the ratios got closer to 1.618.

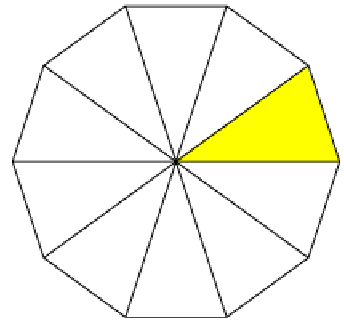
It is called Φ or The Golden Proportion

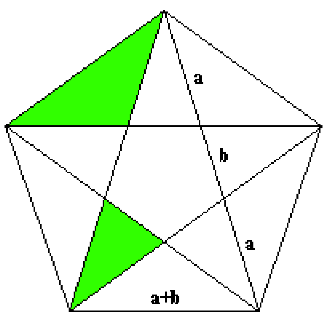
$\Phi = \dfrac{1 + \sqrt{5}}{2} \approx 1.618$

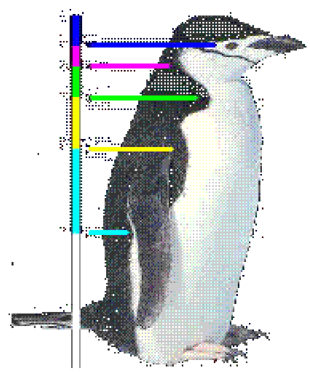
The Golden Proportion:

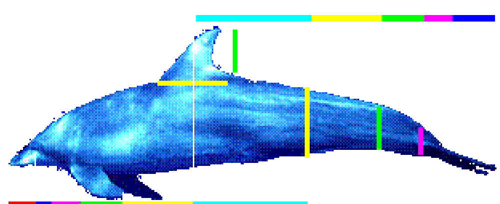


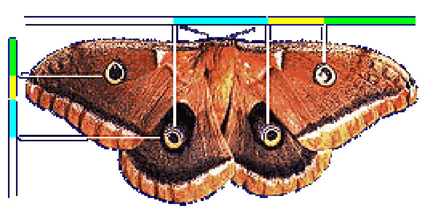












Fibonacci Spiral

also called Equiangular Spiral or Logarithmic Spiral

Shows up in art and nature

any radius crosses the spiral at the same angle

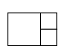
Making an Equiangular Spiral using squares:

Use Fibonacci numbers for the sizes of the squares.

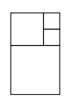
Start with one square

Add a second square

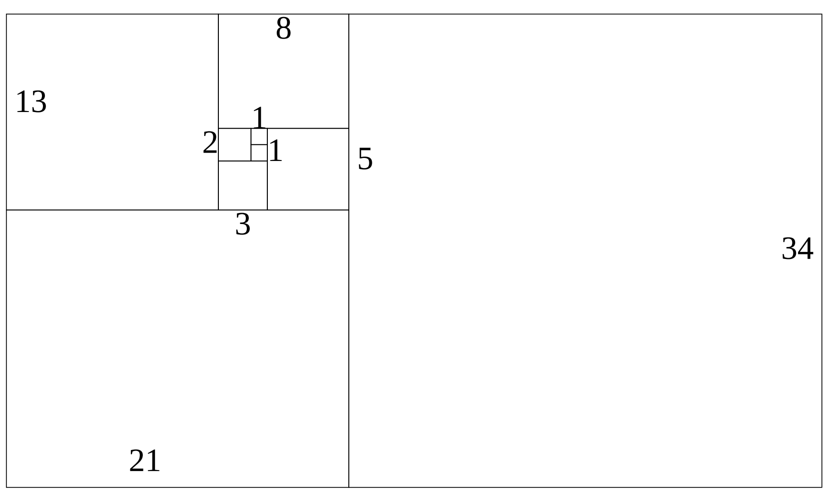
Add a third square, but larger



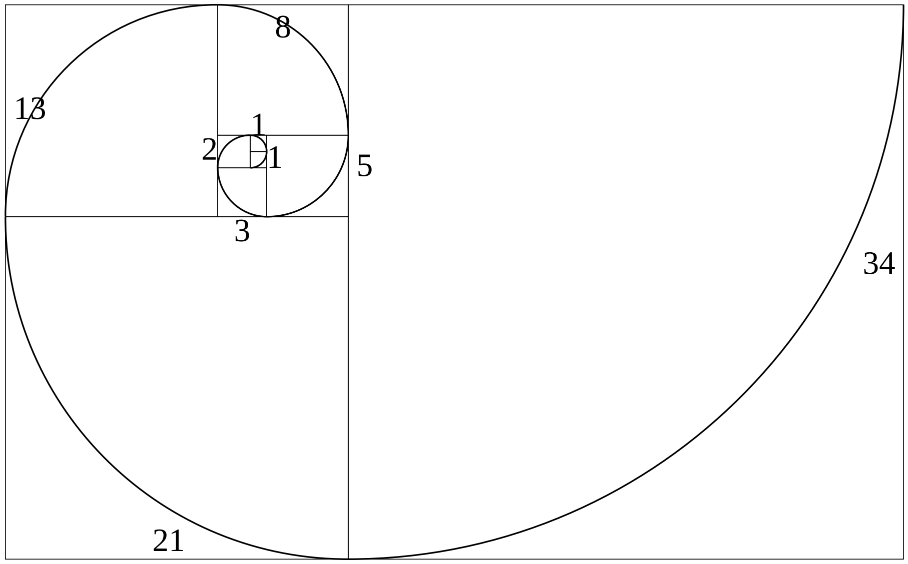
Add another larger square

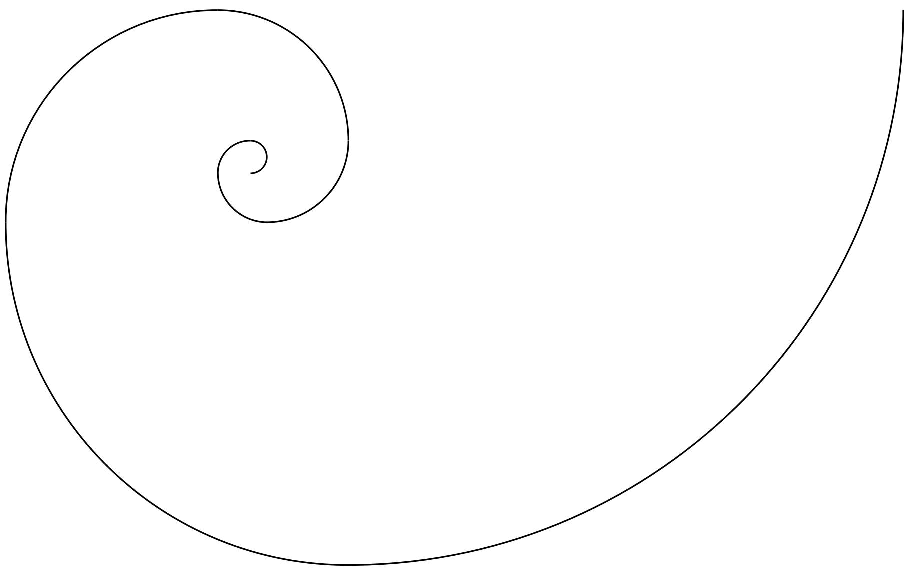


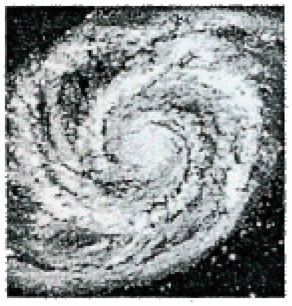
Keep adding!



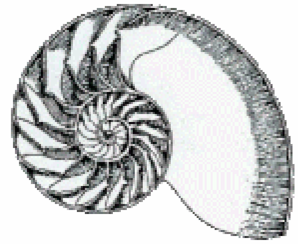
Now, make a spiral. Your start is the bottom, left corner of the first box Draw the spiral by passing through the corners of the squares.







Galaxy



Nautilus shell